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ON SOME NONLINEAR ELLIPTIC AND PARABOLIC PROBLEMS WITH GENERAL MEASURE DATA

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ON SOME NONLINEAR ELLIPTIC AND PARABOLIC PROBLEMS WITH GENERAL MEASURE DATA

Ph.D Thesis

by

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Dédicace

A mes chers parents
Nulle dédicace ne peut exprimer ma profonde affection, mon immense
Gratitude pour tous vos sacrifices, vos conseils et vos prières.
A mon cher petit frère
Que Dieu te prête bonne santé et longue vie.
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Contents

Abstract

The research works that I conducted since the beginning of my PhD were concerned with several tightly related topics, unified mainly by the common regularizing tools used to approach the problems. All of them were devoted to "solving" partial differential equations (PDE's). Must of these problems are nonlinear evolution equations governed by differential divergence operators with measures as data in $\mathcal{M}_b(Q)$. This includes and generalizes various classical problems such as scalar conservation laws, porous medium or Leray-Lions kind problems including a sum of different operators. Many of the problems I considered should be seen as singular version of more elliptic and parabolic problems. I also analyzed some generalized porous medium equations and some nonlinear inequalities. My main activity is the study of relevancy of different solution concepts, it usually leads to results on existence, uniqueness and structural stability of the appropriately defined solutions of these problems. While the methods of resolution using the "cut-off test functions" were often already well established. I treated in a number of works the questions of comportment of the singular part of measure, compactness results, or the asymptotic behavior of solutions u_n as n tends to infinity. Most of the problems under study are of rather academic character, thoroughly motivated by applications from intelligent fluids, continuum mechanics, population dynamics, image processing and electrorheology, etc. For these problems, I develop an approximation techniques and the related "convergence results" using the functional-analysis tools, with a focus on decomposition of measures, convergence of truncates and on coupling of capacities with a priori estimates. These techniques permitted to prove convergence of solutions of several academic and applied problems.

Key words : Quasi-linear elliptic equations ; Nonlinear parabolic equations ; Renormalized solutions ; Weak solutions ; Entropy solutions; Cut-off functions ; Test functions; Truncations ; Auxiliary functions ; mollifying Kernel ; Leray-Lions operators ; Radon measures ; $P-$capacities ; Relative capacities ; Generalized $p(\cdot)-$capacities ; Strong convergence of truncations ; Marcinkiewicz spaces ; bounded domain ; Generalized Lebesgue-Sobolev spaces ; A priori estimates ; Gagliardo-Nirenberg estimates ; Capacitary estimates ; Porous medium equations ; Absorption type equations.

AMS classification : 35R06 ; 32U20 ; 46E30 ; 28A12 ; 35A23 ; 35Q35.

Résumé

Les travaux de recherche que j'ai mené depuis le début de ma thèse étaient dédiés à une série de questions proches les unes des autres, essentiellement reliées par des outils de régularisation communs utilisés dans l'approximation des problèmes, et visant toutes la "résolution" des équations aux dérivées partielles (EDP's). La plupart de ces problèmes sont des équations d'évolution non linéaires gouvernées par des opérateurs différentiels (divergentielles) avec des données mesures dans $\mathcal{M}_b(Q)$. Ceci concerne et généralise plusieurs problèmes classiques tels que les équations des lois de conservation, du milieux poreux, ou du type Leray-Lions faisant intervenir une somme de différents opérateurs. Plusieurs de ces problèmes doivent être considérés comme des versions singulières de plusieurs problèmes elliptiques et paraboliques. Mon activité principale était d'étudier la pertinence des différentes notions de solution; les résultats obtenus peuvent alors conduire à l'établissement de l'existence, de l'unicité et de la stabilité structurelle des solutions définies d'une façon bien adaptée à ces problèmes. Alors que les méthodes de résolution utilisant des "fonctions test isolées" étaient la plupart du temps déjà bien établies, je me suis intéressé dans une série de travaux au comportement de la partie singulière de la mesure, les résultats de compacité, ou le comportement asymptotique des solutions approchées u_n quand n tend vers l'infini. Les problèmes que j'ai étudié, bien que souvent de caractère académique, ont toutefois été, à l'origine, fortement motivés par des applications provenant des domaines de la mécanique des fluides, du traitement d'images, de la dynamique des populations et de l'électrorhéologique, etc. Pour certains de ces problèmes, j'ai participé au développement des techniques d'approximation et des résultats de convergence associés utilisant des outils d'analyse fonctionnelle, en mettant l'accent sur la décomposition de la mesure, la convergence des fonctions troncatures et sur la liaison entre la notion de capacité avec les estimation à priori. Ces techniques ont permis de démontrer la convergence des solutions pour divers problèmes académiques et appliqués.

Mots-clés : Équations elliptiques quasi-linéaires ; Équations paraboliques non-linéaires ; Solutions renormalisées ; Solutions faibles ; Solutions entropiques ; Fonctions isolées ; Fonctions test ; Troncature ; Fonctions auxiliaires ; Suites régularisantes de Kernel ; Opérateurs de Leray-Lions ; Mesures de Radon ; $P-$capacités ; Capacités relatives ; $P(\cdot)-$capacités généralisées ; Convergence forte de troncature ; Espaces de Marcinkiewicz ; Domaine borné ; Espaces de Lebesgue-Sobolev généralisés ; Estimation à priori ; Estimations de Gagliardo-Nirenberg ; Estimation de capacité ; Équations en milieux poreux ; Équations avec terme d'absorption.

Classification AMS : 35R06 ; 32U20 ; 46E30 ; 28A12 ; 35A23 ; 35Q35.

List of Publications

The publications that constitute the basis of the Ph.D. thesis can be found in

Published works and works to appear

[AA1] M. Abdellaoui, M. Kbiri Alaoui, E. Azroul, Existence of renormalized solutions to quasilinear elliptic problems with general measure data, E. Afr. Mat. **29** (2018), 967–985.

[AA2] M. Abdellaoui, E. Azroul, Renormalized solutions for nonlinear parabolic equations with general measure data, Electron. J. Differential Equations, Vol. 2018, No. **132**, pp. 1–21.

Submitted works

[AA3] M. Abdellaoui, E. Azroul, S. Ouaro, U. Traoré, Nonlinear parabolic capacity and renormalized solutions for PDEs with diffuse measure data and variable exponent, Submitted.

[AA4] M. Abdellaoui, E. Azroul, Nonlinear parabolic equations with soft measure data, Submitted.

[AA5] M. Abdellaoui, E. Azroul, H. Redwane, Nonlinear parabolic equations of porous medium type with unbounded term and general measure data, Submitted.

[AA6] M. Abdellaoui, E. Azroul, Non-stability result of entropy solutions for nonlinear parabolic problems with singular measures, Submitted.

[AA7] M. Abdellaoui, E. Azroul, H. Redwane, Renormalized solutions to nonlinear parabolic problems with blowing up coefficients and general measure data, Submitted.

Preprints and works in final phase of preparation

[AA8] M. Abdellaoui, E. Azroul, Orlicz capacities and application to some existence questions for parabolic PDE's having measure data.

[AA9] M. Abdellaoui, E. Azroul, Nonlinear parabolic capacity and renormalized solutions for equations with diffuse measure and exponent variable.

[AA10] M. Abdellaoui, E. Azroul, Asymptotic behavior of renormalized solutions to parabolic equations with measure data and $G-$convergence operators.

[AA11] M. Abdellaoui, E. Azroul, Renormalized solutions to fractional parabolic problems with L^1-data.

List of Symbols

Notations

$*$	convolution product.
C	positive constant which may change line to line.
$\mathbb{R}$	real line.
Ω	open bounded subset of $\mathbb{R}^N$.
∇	gradient of a scalar field $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$.
$p(\cdot)$	variable exponent.
$\mathbb{R}^N$	$N-$dimensional Euclidean space.
k, n	positive integers.
a.e	almost everywhere.
$\partial\Omega$	boundary of the set Ω.
Δu	Laplacian of u.
$d\sigma$	surface measure in $\partial\Omega$, also denoted $\mathcal{H}^{N-1}$.
D^α	$(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}})$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$.
δ_{ij}	symbol of Kronecker.
δ_{x_0}	Dirac mass en x_0.
$\langle \cdot, \cdot \rangle$	scalar product of $\mathbb{R}^N$, duality between X and X'.
$\Vert u \Vert_X$	norm of the vector u in the space X.
u^+, u^-	$\max(u, 0)$, $\max(-u, 0)$.
$\lambda \perp E$	λ is concentrated on a set E such that $\mathrm{cap}_p(E) = 0$.
$\rho_{p(\cdot)}(u)$	convex modular.
$t \in (0, T)$	time's variable, $t > 0$.
$p' = \frac{p}{p-1}$	conjugate exponent of $p > 1$, $(\frac{1}{p} + \frac{1}{p'} = 1)$.
$p^* = \frac{pN}{N-p}$	critical Sobolev exponent of $p < N$.
$\partial_i u = \frac{\partial u}{\partial x_i}$	partial derivative of the function u with respect to the variable x_i.
$\partial_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$	second-order partial derivative with respect to x_i and x_j.
$\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}$	first partial derivatives.
$\partial_j u$ or $D_j u$	partial derivative $\frac{\partial u}{\partial x_j}$ of u in the direction x_j.
$p - \mathrm{cap}(B, \Omega)$	$p-$capacity of the set B in the space Ω.
$\mathrm{meas}(\Omega)$, $\vert\Omega\vert$	measure of the set Ω.
$Q = (0, T) \times \Omega$	parabolic cylinder.
$\Sigma = (0, T) \times \partial\Omega$	parabolic lateral boundary.
$\alpha = (\alpha_1, ..., \alpha_n)$	an $n-$tuples or a multi-index.
$x = (x_1, \cdots, x_N)$	an $n-$tuples of real numbers of $\mathbb{R}^N$.
$\partial^\alpha u = D^\alpha u = \frac{\partial^{\vert\alpha\vert} u}{\partial x^\alpha}$	derivatives of u of order $\vert\alpha\vert$ where α is a multi-index with $\vert\alpha\vert \leq k$.
$dx = dx_1 dx_2 \cdots dx_N$	Lebesgue measure in Ω.
$\Delta_p u = \mathrm{div}(\vert\nabla u\vert^{p-2} \nabla u)$	$p-$Laplace operator.
$\Delta_{p(\cdot)} u = \mathrm{div}(\vert\nabla u\vert^{p(x)-2} \nabla u)$	$p(x)-$Laplace operator.
$\mathrm{Supp}\ u = \overline{\{x \in U : u(x) \neq 0\}}$	support of the function u.

Functional spaces and norms

Let u be a measurable function in Ω and $1 \leq p < \infty$. Let V Banach space and $u : [0,T] \to V$ measurable. Then

$C(\Omega)$	space of continuous (real-valued) functions on Ω with the norm $\|f\| = \sup_{x\in\Omega}\|f(x)\|$.
$\mathcal{D}(\Omega)$	class of all infinitely differentiable functions on Ω with compact support endowed with inductive limit topology.
$\mathcal{D}'(\Omega)$	dual space of $C_0^\infty(\Omega)$ (Distribution space).
$C_0(\Omega)$	class of all continuous functions on Ω that vanishes at boundary.
$H^k(\Omega)$	space $W^{k,2}(\Omega)$.
$H_0^k(\Omega)$	space $W_0^{k,2}(\Omega)$.
$C^k(\Omega)$	class of $k-$times continuously-differentiable functions on Ω $(k \geq 1)$.
$C^{k,\gamma}(\Omega)$	class of functions in $C^k(\Omega)$ whose $k-$th partial derivatives $(k \geq 0)$ are Hölder continuous with exponent γ.
$C^\infty(\Omega)$	class of infinitely differentiable functions on Ω endowed with topology of uniform convergence on compact sets (smooth functions).
$C^\infty(\overline{\Omega})$	class of $C^\infty(\Omega)$ functions such that all its derivatives can be extended continuously to $\overline{\Omega}$.
$C_0^\infty(\Omega)$	class of all infinitely differentiable functions on Ω with compact support.
$\mathrm{Lip}(E)$	space of all Lipschitz functions on E.
$L^p(\Omega)$	space of functions for which the $p-$th power of the absolute value is Lebesgue integrable in Ω for the measure dx, $\|f\|_p = (\int_\Omega \|f(x)\|^p dx)^{\frac{1}{p}}$.
$L^{p(\cdot)}(\Omega)$	space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\rho_{p(\cdot)}(u) < \infty$.
$L^\infty(\Omega)$	space of measurable functions such that $\|u(x)\| < C$ a.e. $x \in \Omega$.
$S_{p(\cdot)}(E)$	admissible test functions for the Sobolev capacity of E.
$W^{1,p}(\Omega)$	Sobolev space $\{u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N\}$, $\|u\|_{1,p} = (\|u\|_p^p + \|\nabla u\|_p)^{\frac{1}{p}}$.
$W^{-1,p'}(\Omega)$	dual space of $W_0^{1,p}(\Omega)$, also denoted $(W_0^{1,p}(\Omega))'$.
$W_0^{1,p}(\Omega)$	closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.
$W^{1,p(\cdot)}(\Omega)$	$= \{u \in L^{p(\cdot)}(\Omega),\ \nabla u \in (L^{p(\cdot)}(\Omega))^N\}$.
$\mathcal{T}^{1,p(\cdot)}(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $T_k(u) \in W^{1,p(\cdot)}(\Omega),\ \forall k > 0\}$.
$\mathcal{T}_0^{1,p(\cdot)}(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $T_k(u) \in W_0^{1,p(\cdot)}(\Omega),\ \forall k > 0\}$.
$C^\infty(Q)$	space of smooth functions with compact support in Q.
$C^k(Q)$	space of compactly supported $k-$times continuously differentiable functions in Q with the norm $\|u\| := \sum_{\|\alpha\|\leq k} \|D^\alpha u\|_\infty$.
$\mathcal{M}^\gamma(Q)$	Marcinkiewicz spaces $\{u : Q \to \mathbb{R},\ u$ is measurable such that meas$\{(t,x) \in Q :\ \|u\| > k\} \leq ck^{-\gamma}$ with $\gamma > 0\}$.
$L^p(0,T;V)$	$= \begin{cases} u : (0,T) \to V \text{ measurable, } \|u\|^p_{L^p(0,T;V)} = \int_0^T \|u(t)\|_V^p dt < \infty, \\ u : (0,T) \to V \text{ measurable, } \|u\|_{L^\infty(0,T;V)} = \underset{t\in[0,T]}{\text{ess sup}}\|u(t)\|_V < \infty. \end{cases}$
$\mathcal{D}([0,T];V)$	space of continuously-differentiable functions with compact support in $[0,T]$.
$C^k([0,T];V)$	space of $k-$times $(k \geq 1)$ continuously-differentiable functions on $[0,T] \to V$.
$\mathcal{M}_0(\Omega), \mathcal{M}_0(Q)$	space of bounded Radon measures in $\mathcal{M}_b(Q)$ which does not charge sets of null capacity.
$\mathcal{M}_s(\Omega), \mathcal{M}_s(Q)$	space of all singular measures on Ω, Q with respect to the capacity.
$\mathcal{M}_b(\Omega), \mathcal{M}_b(Q)$	linear space of finite signed measures (or Radon measures) on Ω, Q.

Functions and intervals

$v+ = \max(v,0)$	positive part of v.
$v- = -\min(v,0)$	negative part of v.
$p(\cdot) : \Omega \to [1,+\infty)$	variable exponent.
$\mathrm{sign}(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$	sign function.
$G_k(v) = (\lvert v\rvert - k)_+ \mathrm{sign}(v)$	level set function.
$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$	characteristic function.
$T_k(v) = v - G_k(v) = \max\{-k, \min\{k,v\}\}$	truncation function.
$\{\lvert v(t)\rvert > k\} := \{x \in \Omega : \lvert v(t,x)\rvert > k,\ t \in [0,T]\}$	set where $v(t,x)$ is positive.
$\{\lvert v(t)\rvert < k\} := \{x \in \Omega : \lvert v(t,x)\rvert < k,\ t \in [0,T]\}$	set where $v(t,x)$ is negative.

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Avant Propos

Dans le domaine des EDP's et de la recherche de la solution beaucoup de travaux sont focalisés sur le cas elliptique à données mesures. Les modèles des EDP's "classiques" définissent l'importance de la notion de capacité par rapport à la décomposition de la donnée en utilisant des mesures tel que la mesure diffuse ou singulière. Celles-ci déterminent l'importance de la décomposition en fonction de l'apparition des termes définis dans le problème approché. Cependant ces méthodes ne permettent pas de vérifier si les solutions délivrées par le problème sont uniques, ou, si les nouveaux termes tel que le "terme d'absorption" sont bien définis. Nous traitons dans cette étude de thèse uniquement les travaux avec des données mesures. L'idée c'est de retrouver, à l'aide de plusieurs travaux récents, des nouvelles approches sur les problèmes en question pour ensuite les extraire et les structurer dans un cas plus général. Il faudra donc mettre à jour des approximations adaptées lorsque des nouveaux termes apparaissent. Nous devons pour cela déterminer :

- Quelles écritures de solutions permettant d'obtenir des meilleures approximations du problème pour retrouver les solutions dans le problème initial.
- La méthode permettant d'établir, et d'estimer, le lien entre les solutions contenues dans le problème approché et les solutions du problème de base.

Cette problématique mathématique s'inscrit donc parfaitement dans la problématique posée dans cette thèse avec différentes phases que l'on peut retrouver comme le pré-traitement des cas simples, la recherche des nouvelles extensions et l'extraction des nouvelles questions. La suite de ce rapport citent les contributions scientifiques principales qui ont été apportées jusqu'à aujourd'hui par différents auteurs et contient les points que nous avons traités et enfin quelques extensions et les perspectives des travaux à venir (problèmes ouverts).

Une étude approfondie sur les équations aux dérivées partielles quand le deuxième terme est une mesure mène à un article de *A. Prignet* en 1999 avec *G. Dal Maso, F. Murat, L. Orsina*, *"Renormalized solutions for elliptic equations with general measure data"*, permet de dégager des idées intéressantes quand la mesure est décomposée en un terme absolument continu et un autre terme singulier, la difficulté lors du traitement de ce type de problèmes quasi-linéaires ou même non-linéaires est le terme singulier qui est concentré sur une partie de capacité zéro, une technique est nécessaire pour remédier à cette difficulté est d'introduire des fonctions isolées qui permettent d'avoir des convergences adaptées dans les problèmes approchés et on pourra donc faire disparaître le terme singulier et trouver une solution à notre problème initiale par passage à la limite. Il faut noter que lors du traitement de ce type de problèmes il faut toujours faire appel à la notion de troncature et la notion de $p-$capacité trouvé par l'inférieur de certaines fonctions admissibles, généralisées par la suite dans le cas Sobolev avec exposant variable, la notion de $p-$capacité joue un rôle important dans la théorie du potentiel et utilisée pour mesurer les propriétés finies des fonctions et des parties, cela nous mène à la notion de quasi-partout et quasi-continue. Enfin il faut mentionner que les techniques usuelles d'approximation et d'estimation restent valables et qui seront adaptées et utilisées pour trouver les convergences désirées. Après une lecture des approches existantes dans l'article de *A. Prignet*, nous avons participé activement au développement d'une démonstration du problème quasi-linéaire elliptique avec une mesure de variation totale en collaboration avec Pr. *Mohammed Kbiri Alaoui* [1], nous avons pu démontrer un résultat d'existence et de stabilité du problème, qui permet notamment la généralisation au cas non-linéaire avec un terme gradient, un terme fortement non-linéaire ou même un graphe. Cette démonstration a été développée au moyen des techniques de compacité et d'approximation. La version dont nous disposons est générale et les premiers résultats sont encourageants puisque nous parvenons à générer le premier modèle traité dans le cas exposant variable avec une mesure

[1]King Khalid University, Abha, Saudi Arabia

générale. En outre nous avons travaillé à l'extension de notre outil de travail à d'autres modèles plus généraux tels que: le cas des équations parabolique faisant intervenir des opérateurs non-linéaires contenant un terme mesure. Afin de compléter l'architecture finale correspondant à notre sujet de thèse, nous avons commencé l'étude de la relation entre la capacité parabolique et la mesure qui consiste à la compréhension de la notion de $p(\cdot)-$capacité (capacité généralisée) produite dans les nouveaux articles soumis de *S. Ouaro* afin d'obtenir un outil permettant l'analyse du terme mesure dépendant du temps. Il y a deux difficultés à surmonter dans cette opération :

- La première est liée à la représentation de la mesure qui dépend du temps.
- La seconde consiste à l'extension de la démonstration du cas elliptique afin d'intégrer le terme $\frac{\partial u}{\partial t}$ dans le modèle.

Concernant le 1^{er} point, nous avons utilisé les travaux de *F. Petitta* qui est un standard référence reconnu dans les problématiques de la mesure qui dépend du temps. Nous avons réalisé une première conception et une large documentation a été générée automatiquement après la lecture de quelques travaux dans ce sens. Concernant le $2^{\text{ème}}$ point, cela a demandé une étude approfondie du cas parabolique et fait parti des travaux réalisées. Enfin, nous avons pu donner une référence de base à large problèmes. Cela grâce aux travaux de *A. Prignet*, *A. Porretta*, *A. Malusa*, *F. Murat*, *E. Azroul*, *S. Ouaro*, *M. Sanchon*, *C. Zhang* et autres. Les articles de bases comme [**DPP, DP**], et l'article [**Pe1**] constituent des ressources de base pour le cas parabolique avec mesure, l'avancement de ce titre a été renforcé par différents auteurs comme *M.-F Bidon*, *J. Droniou*, *H. Redwane* etc..., tous ces travaux concernent les espaces de Sobolev classiques ($p = cte$), grâce aux travaux de *U. Traoré* qui a pu donner une généralisation du théorème de décomposition de la mesure dans le cas exposant variable dans son article *"$p(\cdot)-$parabolic capacity and decomposition of measure"*, nous avons pu avancer dans l'étude du cas parabolique. Les principaux points traités dans ce cas étaient de donner une définition adéquate des solutions renormalisées dans le cas exposant variable, les propriétés de ces solutions, afin d'utiliser l'argument du théorème du convergence forte des troncatures pour montrer l'existence et l'unicité, tout en utilisant la référence [**DPP**], le travail utilise beaucoup d'arguments inspirés du cas elliptique, notre but était de donner dans un premier temps une approximation adéquate de la mesure $\mu \in \mathcal{M}_0(Q)$, un terme qui dépend du temps apparaît dans la décomposition donc il fallait le rajouter à la solution et étudier le problème avec changement de variable $u - g_t$, quelques difficultés apparaissent dans la preuve qui étaient surmontées à l'aide des outils de base (inégalité de Hölder généralisée, Log-Hölder Continuité,...etc), les estimations a priori obtenues sur la solution u ou sur le terme $v = u - g_t$ permettent de dégager des convergences adaptées lors du passage au problème initial, il faut noter que dans ce travail l'approche de renormalisation est appliquée sur le variable $u - g$ non pas sur u, notons que cette méthode ne peut pas s'appliquer à des équations avec terme d'ordre inférieur $h(u)$ en remplaçant $h(u)$ par $h(v + g)$ (voir chapitre 5) sans avoir une condition de bornitude sur g qui apparait dans la décomposition de μ. A l'heure actuelle une extension est possible en changeant la démonstration et en inspirant de l'article [**Pe3**] de *F. Petitta*. A partir de l'idée d'article [**Ma**] de *A. Malusa*, nous avons pu réaliser aussi une extension du cas classique fourni par [**Pe1**], l'avantage majeur de ce travail consiste sur le fait de passer à la limite dans le problème approché utilisant la convergence presque partout du gradient dans Q. Des nouveaux travaux de *F. Petitta*, *A. C. Ponce*, *A. Porretta* sur la notion de solutions renormalisées permettant de montrer l'existence et l'unicité pour une large classe de problèmes, et montrant qu'il est tout à fait possible d'améliorer les réponses apportées par les résultats basiques de compacité, ces nouveaux articles [**PPP1, PPP2**] utilisent la notion des mesures équi-diffuses et traitent juste le cas des mesures diffuses, pour le cas générale, la notion a été introduite dernièrement par *F. Petitta* dans son travail [**Pe3**]. L'application de cette méthode à l'étude de quelques problèmes d'écoulement en milieu poreux est nouvelle, l'intérêt de chercher l'existence des solutions en remplaçant le terme u_t par $b(u)_t$ (qui peut dépendre aussi de x), les méthodes classiques dans ce sens ont été appliquées par différents auteurs à différents problèmes techniques et physiques. Afin de traiter ce cas nous avons proposé des nouvelles approximations de la mesure comme indiqué dans [**PPP1**] imposant quelques conditions sur la fonction b. On a pu prolongé cette étude à des équations générales, on a eu affaire à des termes $b(x, u)$ bien plus difficile à traiter en collaboration avec Pr. *Hicham Redwane*[2]. Malgré tout, ces méthodes de résolution qui déterminent l'existence de la solution par des approximations, exigent des conditions sur a et μ, exigent aussi des longs calculs et ne permettent pas de montrer l'unicité dans certains cas. Il faudrait pour cela chercher à rendre ces méthodes plus rapides, compatibles avec un usage générale (espaces plus générales).

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Enfin notons qu'il est tout à fait possible d'améliorer les réponses apportées à ces résultats aux problèmes avec des termes qui explosent ou avec des espaces de type modulaire. Deux des points fondamentaux du sujet sont encours d'exploitation :

- L'amélioration de l'approximation du problème contenant un terme d'absorption dans le cas exposant variable.
- Application de ces méthodes en cas anisotropique et obtenir des résultats satisfaisantes dans le cas des espaces d'Orlicz-Sobolev ou Musielak-Orlicz-Sobolev.

Quelques problèmes ouverts seront proposés à la fin de ce rapport.

Introduction

This thesis is devoted to the study of a class of nonlinear elliptic and parabolic initial boundary value problems with measure data, in bounded domains. If $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded open set, let $A(u) = -\mathrm{div}(a(x, \nabla u))$ be an operator acting from the space $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p'(\cdot)}(\Omega)$, $p_- > 1$, and satisfying the Leray-Lions assumptions (see (2.2.2)–(2.2.4) below) which imply appropriate coercivity and monotonicity properties. We study, under suitable hypotheses, the existence and the asymptotic behavior of solutions of initial boundary problems of the type

$$(1) \qquad \begin{cases} A(u) = \mu & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a general bounded Radon measure on Ω, $p(\cdot) : \Omega \mapsto \mathbb{R}^N$ is a measurable function such that

$$(2) \qquad \begin{gathered} \exists C > 0 : \quad |p(x) - p(y)| \leq \frac{C}{-ln|x-y|}, \quad \text{for} \quad |x-y| < \frac{1}{2}; \\ 1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x) < N. \end{gathered}$$

To fix the ideas, one can consider, as a special example of (1), the $p(x)-$Laplace initial boundary value problem

$$(3) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu & \text{in } \ \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If both A and μ depends on time, then A is generalized to the case of parabolic pseudo-monotone operators satisfying the natural extensions of the classical Leray-Lions assumptions acting from $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$ into its dual space $L^{p'_-}(0, T; W^{-1,p'(\cdot)}(\Omega))$. In this case, a whole theory was recently developed about the $p(\cdot)-$parabolic capacity for the parabolic problems whose model is

$$(4) \qquad \begin{cases} u_t - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\mu \in \mathcal{M}_0(Q)$ (measures which do not charges sets of zero $p(\cdot)-$capacity) and $u_0 \in L^2(\Omega)$ is a smooth initial data. This difficulty can be overcome by defining the solution in renormalized sense, by adapting the techniques of non-constant case. With slightly modifications one can investigate the asymptotic behavior of a sequence of approximate sequence of renormalized solutions u_n as n goes to infinity, proving that it converges, in a suitable way, to the solution of the same problem, that is the solution of the parabolic boundary value problem

$$(5) \qquad \begin{cases} u_t - \mathrm{div}(a(t,x,u,\nabla u)) = \mu & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

The difficulties in the study of such problems concern the possibly very singular right hand side that forces the choice of a suitable formulation that ensures both existence and uniqueness of the solution.

Under the classical assumptions that p is constant and μ is bounded measure, the existence of distributional solution was proved in **[BG1]**, by approximating (5) with problems having regular data and using compactness arguments. But, due to the lack of regularity of the solution, the distributional formulation is not strong enough to provide uniqueness (see counterexample of J. Serrin [**Ser**]), as it can be proved by restricting the set of admissible functions.

In the case of linear operators the lack of uniqueness can be overcome by defining the solution in a duality sense, and then adapting the techniques of the stationary case introduced in [**S**] (see Section 1.5). However, for nonlinear operators a new concept of solution is necessary to get a well-posed problem. In the case of problem (5) with $\mu \in L^1(\Omega)$, this was done independently in [**B6**] and in [**Dal**], where the notions of *renormalized* solution, and of *entropy* solution, were respectively introduced (see Sections 1.7 and 1.8). Both these approaches allow to obtain existence and uniqueness of solutions. Unfortunately, these definitions do not extend directly to the case of a general, possibly singular, measure μ. In [**BGO1**] the authors extend the result of existence and uniqueness to a larger class of measures which includes the L^1-case. Precisely, they prove (in the framework of renormalized solutions) that problem (5) has a unique solution for every measure μ which does not charge the sets of null $p-$capacity (see Section 1.4).

Under some assumptions on a, If $\mu \in L^{p'}(Q)$ the existence and uniqueness of a weak solution u of (5) belonging to suitable energy space and to $C([0,T]; L^2(\Omega))$ was proved in [**L**]. In case of linear operators the difficulty can be overcome by defining the solution through the adjoint operator, this method is used in [**S**] and yields a formulation having a unique solution. For nonlinear operators, the authors in [**BM**] and [**P**] extend the results in two different directions, assuming that $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, they prove existence of renormalized solution, and of entropy solution, the same notions of solutions are used to ensure existence and uniqueness of equations with bounded Radon measures on Q that does not charge the sets of zero parabolic $p-$capacity (see [**BM, Po1, DPP**]), the authors show in [**DP**] that these two notions of solutions actually turn out to coincide. The importance of the measures not charging sets of null $p-$capacity was first observed in the stationary case. In order to use a similar approach in the non-constant exponent case, the theory of $p(\cdot)-$capacity related to the elliptic operators has been developed in [**HHK**], where the authors also investigated the relationship between measures and capacities.

This concept of capacity is of fundamental importance in the study of solutions of partial differential equations and classical potential theory. For example, a characterization of the relationship between sets and zero parabolic $p-$capacity sets is fundamental. In the stationary case, capacity is related to the underlying Sobolev space, but the situation is more delicate for parabolic partial differential equations. Indeed the theory of capacity seems to be related more closely to the existence and uniqueness of the solution of some elliptic and parabolic problems. When $p = 2$, the thermal capacity related to the heat equation, and its generalization have been studied by Lanconelli [**Lanco**] and Watson [**Wat**]. Capacities defined in terms of functions spaces are introduced in [**Aro, EP, HP, P, Zie**]. For non-quadratic case, the authors in [**DPP**], as well as Saraiva [**Sar**], introduced and studied the notion of parabolic capacity to get a representation theorem for measures that are zero on subsets of Q of null capacity (see Section 1.12).

Thanks to a decomposition result (Proposition 2.4 below) proved in [**Zha**], if μ is absolutely continuous with respect to the $p(\cdot)-$capacity one can still set problem (1) in the framework of renormalized solutions, as in the Lebesgue-Sobolev spaces case, the idea formally consists in the use of test functions which depends on the solution itself. Thus, the definition of renormalized solution of problem (1) can be extended to the case of general measure μ by adapting the idea of [**DMOP**]. Notice that the notion of renormalized solution was introduced by DiPerna and Lions [**DL1**] for the study of Boltzmann equations and [**DL2**] for Fokker-Planck-Boltzmann equations. It was then adapted to the study of some nonlinear elliptic, parabolic and evolution problems in fluid mechanics, see [**LM, BGDM, BMR, BR**]. Here we extend the notion of renormalized solution for general measure data μ and so, this notion will turn out to be coherent with all definitions of solution given before for problems (1) and (5) . One of essential results (Theorem 3.13 below), gives a generalization of a decomposition result using the $p(\cdot)-$parabolic capacity developed in [**OT**]. This extends Theorem 2.28 in [**DPP**]. In this thesis we prove also the existence and uniqueness of renormalized solutions to the parabolic problems (4) for arbitrary $\mathcal{M}_0(Q)-$data using compactness results.

In Chapter 1 we first recall some basic tools and preliminary results concerning the theory of both elliptic and parabolic differential problems with measure data, we will state a generalized version of Lebesgue and Sobolev spaces and a useful existence results contained in [**DMOP**] and [**Pe1**]; moreover we introduce the notations we will use throughout rapport of thesis.

Chapter 2 deals with the case of Dirichlet problem in divergence form with Radon measure μ with bounded total variation on Ω and variable growth, proving the existence of a special type of distributional solutions, the so-called "renormalized solutions" under the Log-Hölder assumption (2) of quasi-linear elliptic problems

$$(6) \qquad \begin{cases} -\operatorname{div}(a(x,\nabla u)) = \mu & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the operator $u \mapsto \operatorname{div}(a(x,\nabla u))$ is a classical monotone operator from $W_0^{1,p(\cdot)}(\Omega)$ into $W^{-1,p'(\cdot)}(\Omega)$, and μ belongs to $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures on Ω. In this part, we will present the definition of renormalized solution and recent results taken from our joint work with Mohammed Kbiri Alaoui. Of course, in this general case (which includes the problem with $\mu_s = 0$), solutions will not found in $W_0^{1,p(\cdot)}(\Omega)$ but in a large Sobolev space, namely $W_0^{1,q(\cdot)}(\Omega)$ for every $q(\cdot) < \frac{N(p_- - 1)}{N-1}$ if $p_- > 2 - \frac{1}{N}$ (since $\frac{N(p_- -1)}{N-1} > 1$ if and only if $p_- > 2 - \frac{1}{N}$). For smaller values of $p(\cdot)$, solutions may even not belong to $L^1(\Omega)$ and we need to use the functional class $\mathcal{T}_0^{1,p(\cdot)}(\Omega)$ (see Section 1.1). Thus in Theorem 2.9 we prove that if μ belongs to $\mathcal{M}_b(\Omega)$ (space of all signed measures on Ω, i.e., $\mu = \mu_0 + \mu_s$) and a satisfies (2.2.2) $-$ (2.2.4) then there exist a renormalized solution u of (6). This result is also contained in **[AA1]**.

In Chapter 3 we study the problem of finding solutions of (7) for every measure μ, and in particular the link between the parabolic $p(\cdot)-$capacity and the measure μ which is needed to have existence of solutions. To simplify some technical tools, we deal with the case of absolutely continuous part μ_0 of μ with respect to the $p(\cdot)-$capacity called diffuse measures (i.e., $\mu \in \mathcal{M}_0(Q)$). Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, and let $p(x) : \overline{\Omega} \mapsto [1, +\infty)$ be a continuous, real-valued function (the variable exponent) with $p_- = \min_{x \in \overline{\Omega}} p(x)$. We are interested in the existence and uniqueness of the renormalized solution of parabolic problems whose model

$$(7) \qquad \begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $T > 0$ is any positive constant, $u_0 \in L^1(\Omega)$ is a nonnegative function, $1 < p_- < \infty$, $u \mapsto -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$laplacian operator, and μ is a measure with bounded variation over $Q = (0,T) \times \Omega$ which does not charge the sets of zero $p(\cdot)-$capacity in accordance with Definition 3.10. Moreover we suppose that μ depends on time variable t, we extend the theory of capacity to generalized Sobolev spaces for the study of nonlinear parabolic equations, we introduce the definition and some properties of renormalized solutions and we show that diffuse measures can be decomposed in space and in time. As consequence, we show the existence and uniqueness of renormalized solutions. The used main technical tools include estimates, compactness and convergence results. The contents of this Chapter is a joint result with, respectively, Stanislas Ouaro [3] and Urbain Traoré [3] in **[AA3]**.

Chapter 4 is devoted to the study of the asymptotic behaviour, as ϵ tends to zero, of a sequence of renormalized solutions (u_ϵ) to the problem

$$(8) \qquad \begin{cases} u_t - \operatorname{div}(a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)) = \mu_\epsilon & \text{in } (0,T) \times \Omega, \\ u_\epsilon = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_\epsilon(0) = u_0 & \text{in } \Omega, \end{cases}$$

where (μ_ϵ) is a sequences of measures with splitting converging to μ, and

$$\lim_{\epsilon \to 0} a_\epsilon(t,x,s_\epsilon,\zeta_\epsilon) = a_0(t,x,s,\zeta),$$

for every sequence $(s_\epsilon, \zeta_\epsilon) \in \mathbb{R} \times \mathbb{R}^N$ converging to (s,ζ) and for a.e. $(t,x) \in Q$. Here both a_ϵ and μ are supposed to be dependent on time. We first characterize the measures we consider; indeed, it is easy to see that, if $\mu \in \mathcal{M}_b(Q)$ does depend on time, then $\mu = f - \operatorname{div}(G) + g_t + \mu_s$ with $f \in L^1(Q)$, $-\operatorname{div}(G) \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_t \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\mu_s \perp p-$capacity, that is, thanks to a result of **[FST]**, $\mu = \mu_0 + \mu_s$ with $\mu_0 \in \mathcal{M}_0(Q)$ the space of all bounded Radon measures on Q that not charge the sets of zero parabolic $p-$capacity and $\mu_s \in \mathcal{M}_s(Q)$ the space of all singular measures on Q with respect to the $p-$capacity, to deal with the general case we prove an improved result generalizing a result of **[Ma]** which dealt with elliptic problems. The main point which allows to go further the previous works, is the proof of the almost everywhere convergence of

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gradients in Proposition 4.16 using techniques developed in [**Po1, Pr1**]. Note that to treat the general case, we also prove a technical lemma that involves a compactness argument and in particular a nonlinear convergence result contained in [**DPP**] and we show the interest of cut-off functions to deal with, the possibly singular, measure using the strong convergence of truncates in order to obtain a stability result. All these results are contained in [**AA2**].

In Chapter 5, our approach estimates by regarding solution of some mathematical models of porous media equations obtained through a stability argument in the sense that, letting $\{\mu_n\}$ be the convolution of μ with a regularizing sequence of mollifiers (see Figure 1), we consider the approximating problems of the following model equation

$$\begin{cases} b(u)_t - \operatorname{div}(a(t,x,\nabla u)) = \mu_0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \end{cases} \tag{9}$$

where b is a strictly increasing C^1-function such that $b_0 \leq b'(s) \leq b_1$ for positive constants b_0 and b_1, $b(0)=0$, $a(t,x,\nabla u)$ is a Leray-Lions operator and $\mu_0 \in \mathcal{M}_0(Q)$. With this model in mind, the approach followed in this part is to consider sequences (μ_n) of equidiffuse measures having a special properties. Our strategy will be to associate to every renormalized solution a sequence of parabolic problems solved by its truncations. If u is a solution in the sense of distributions to problem (9) obtained by approximation (in particular if u is a renormalized solution, see Theorem 1.2 in [**PPP1**], then the truncations of u are solutions in the sense of distributions to parabolic problems of the same form with suitable measure data, see [**PPP1, PPP2**]). The key point in the existence result being the proof of the strong compactness of suitable truncations of the approximating solutions in the energy space, we refer to [**AA4**] for more details.

In Chapter 6, whose main issues are contained in a joint work with Hicham Redwane (see [**AA5**]), we give the same type of result of a rather different class of operators. In fact, we study a nonlinear problem whose model is

$$\begin{cases} b(x,u)_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \end{cases} \tag{10}$$

where $1<p<N$, $b(x,u)$ is an unbounded function of u, $b(x,\cdot):\Omega\times\mathbb{R}\mapsto\mathbb{R}$ is a Carathéodory and increasing C^1-function with $b(x,0)=0$, $b(x,u_0)\in L^1(\Omega)$ and there exists λ, $\Lambda>0$, a function $B(x)\in L^p(\Omega)$ such that

$$\lambda \leq \frac{\partial b(x,s)}{\partial s} \leq \Lambda \text{ for a.e. } (x,s)\in\Omega\times\mathbb{R}, \tag{11}$$

$$|\nabla_x b(x,s)| \leq B(x) \text{ a.e. } x\in\Omega, \tag{12}$$

and $\mu\in\mathcal{M}_b(Q)$ is a general, possibly singular, measure dependent on time. In the literature, the divergentiel term assured to have a natural growth since it forces, in some sense, the solution belong to the energy space $L^q(0,T;W_0^{1,q}(\Omega))$ for all $q<p-\frac{N}{N+1}$. These kind of equations, that called generalized porous medium equations, arise from a class of applications in continuum mechanics, population dynamics and image processing, have been largely studied recently, especially for divergence monotone operators $-\operatorname{div}(a(t,x,\nabla u))$ where $a:Q\times\mathbb{R}^N\mapsto\mathbb{R}$ is a Carathéodory function; the assumptions on the nonlinearity a, namely (6.3.15), (6.3.16) and (6.3.17), are rather standard since they ensure, for instance, the existence but not the uniqueness of the solution even with stronger assumptions on a, namely the strong monotonicity and Lipschitz continuity, or the Hölder continuity with respect to the gradient (these assumptions are satisfied, for instance, by the function $a(t,x,s,\zeta)=|\zeta|^{p-2}\zeta$). Actually, the asymptotic result is obtained via a suitable use of approximation result contained in [**PPP2**], and then applying arguments similar to those of Chapter 5. We first prove a capacitary estimate. As we said before, to apply arguments of Chapters 3 and 4, we need to impose a restriction on the decomposition of the datum μ, essentially, the time dependent term g should be bounded (i.e., $g\in L^\infty(Q)$) to handle the case of problems with absorption terms; if μ is a general, possibly singular, bounded Radon measure, we need to prove that a solution exist in the sense of renormalized solutions; this machinery was developed using a new definition and approximation result, with the use of the "near–far from approach" extended in [**Pe3**] for the parabolic case.

In Chapter 7, We try to emphasize the fact that, in the inequality case, the role played by the renormalized solutions is played by the entropy solutions and this definition can be extended to problems with data $\mu = \lambda + g$ taken such that λ is concentrated on a set E of zero $p-$capacity plus a function $g \in L^1(Q)$. Suppose we have a sequence $\{f_n\}$ of functions which converges to λ in the weak–* topology of measures, and a sequence g_n which converges to g in $L^1(Q)$, we prove that non-existence result holds true for the variational inequality

$$\langle u_t - \operatorname{div}(a(t,x,\nabla u)) - \mu, v - u\rangle \geq 0 \tag{13}$$

with $v \in K = \{w \in L^p(0,T;W_0^{1,p}(\Omega)) : |w| \leq 1\}$ for every v in K, we provide a characterization of the solution in terms of approximating sequences of variational inequalities making use a special type of suitable test functions to deal with the singular part of measure. we obtain a nonexistence result consistency with the classical theory of variational inequalities. This result is also contained in the paper [**AA6**].

The Chapter 8 (see [**AA7**]) deals with an approximation result which leads to existence of solution, we introduce the concept of (possibly renormalized) solution in the case of quasilinear parabolic diffusion type equations having continuous coefficients which blow up for a finite value of the unknown with an initial data $u_0 \in L^1(\Omega)$ and a second hand μ in $\mathcal{M}_b(Q)$

$$\begin{cases} u_t - \operatorname{div}(d(u)Du) = \mu & \text{in } (0,T)\times\Omega,\\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x) = u_0 & \text{in } \Omega, \end{cases} \tag{14}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $T > 0$, $d(s) = (d_i(s))_{i=1}^N$ is a diagonal matrix, such that the coefficients $d_i(s)$ are continuous on an interval $]-\infty, m[$ of $\mathbb{R}$ $(m > 0)$ with values in $\mathbb{R}^+ \cup \{+\infty\}$. To achieve the main result it is essential a regularity assumptions in it's coefficients: either a precise condition of coefficient of order p. Then it is proved that if the initial datum u_0 is smaller than the level of the domain of $d_i(s)$ (i.e. $u_0 \leq m$ a.e. in Ω), then $u \leq m$ a.e. in Q, and both $T_k(u)$ and $d(u)DT_k^m(u)\chi_{\{-k<u<m\}}$ satisfy regularity results. It may be considered as the parabolic counterpart of the elliptic framework analyzed in [**BR2, Or**] and the extension of the corresponding parabolic results [**VG2, VG3, ZR**], it should be noted that problems (14) are much more complex (since the definition induces three parameters m, p and truncation-level k), some feature as the regularizing coefficients and singular term are intrinsic of the parabolic setting. The purpose of this Chapter is to exploit, to a certain degree, the a priori estimates and the compactness convergences in order to establish a new existence result which extends in possibly different directions previous results dealing with this question.

The thesis finishes with an Appendix where some known results, open problems and interesting remarks, necessary to the development of this work, are collected.

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CHAPTER 1

A review on some preliminary tools and basic results

1.1. Notations and functional elliptic spaces

We set by $\mathbb{R}^N$ the N-euclidean (simply $\mathbb{R}$ if $N = 1$, while $\mathbb{R}^+ = (0, +\infty)$) on which the standard Lebesgue measure is concentrated, as defined on the $\sigma-$algebra of Lebesgue measurable sets. The scalar product between two vectors a, b in $\mathbb{R}^N$ will be denoted by $a \cdot b$. Given an open bounded subset Ω of $\mathbb{R}^N$, whose boundary will be denoted by $\partial\Omega$, we set by $C_c(\Omega)$ and $C_c^\infty(\Omega)$ the space of continuous, respectively C^∞, functions with compact support in Ω, while $C(\Omega)$ will denote functions which are continuous in the whole closed set $\overline{\Omega}$. We refer to **[Kes]** for the definition of the space of distributions $\mathcal{D}'(\Omega)$, that is the space of continuous linear functionals from $C_c^\infty(\Omega)$ into $\mathbb{R}$.

Considering $C_c(\Omega)$ with the topology of locally uniform convergence, we denote its dual space by $\mathcal{M}(\Omega)$, which is called the space of Radon measures μ, since, by means of Riesz's representation theorem, we will identify the element μ in $\mathcal{M}(\Omega)$ with the real valued additive set function associated, which is defined on the $\sigma-$algebra of Borelian subsets of Ω, and is finite on compact subsets. Thus with $\mu^\pm$ we mean the positive measures, mutually orthogonal, of the Hahn decomposition of μ, that is $\mu = \mu^+ - \mu^-$. We will always deal with the subset of $\mathcal{M}(\Omega)$ consisting of measures μ whose total variation $|\mu| = \mu^+ + \mu^-$ is finite on Ω, that is $|\mu|(\Omega) < +\infty$; this subset of bounded Radon measures is denoted by $\mathcal{M}_b(\Omega)$, while $\mathcal{M}_b^+(\Omega) = \{\mu \in \mathcal{M}_b(\Omega) : \mu \geq 0\}$. The restriction of a measure μ on a subset E is denoted by $\mu \perp E$ and defined as follows:

$$\mu \perp E(B) = \mu(E \cap B) \quad \text{for every Borelian subset } B \subset \Omega. \tag{1.1.1}$$

If (1.1.1) holds true, we will say that μ is concentrated on E.

For $1 \leq p \leq \infty$, we denote by $L^p(\Omega)$ the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that, if $p < +\infty$, $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{\frac{1}{p}} < +\infty$, or which are essentially bounded (with respect to Lebesgue measure) if $p = \infty$. For the definition, the main properties and results on Lebesgue spaces we follow **[Br]**. Given a function u in a Lebesgue space, we set by $\frac{\partial u}{\partial x_i}$ its partial weak derivative in the x_i direction defined in the space of distributions $\mathcal{D}'(\Omega)$ as

$$\langle \frac{\partial u}{\partial x_i}, \varphi \rangle = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx \quad \forall \varphi \in C_c^\infty(\Omega),$$

and we denote by $\nabla u = (\frac{\partial u}{\partial x_i}, \cdots, \frac{\partial u}{\partial x_N})$ the gradient of u defined in this weak sense.

The Sobolev space $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, is the space of functions u in $L^p(\Omega)$ such that ∇u belongs to $L^p(\Omega)^N$ as well (i.e. ∇u is a vector of N functions each belonging to $L^p(\Omega)$), endowed with the norm $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$, while $W_0^{1,p}(\Omega)$ will denote the closure of $C_c^\infty(\Omega)$ with respect to this norm. We still follows **[Br]** for the basic tools related to Sobolev spaces and their main properties. Let us just recall that, for $1 < p < \infty$, the dual space of $L^p(\Omega)$ can be identified with the space $L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p, and that the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$. By a well-known result, each element T in $W^{-1,p'}(\Omega)$ can be written in the form $T = \mathrm{div}(F)$ where F belongs to $L^{p'}(\Omega)^N$.

For every $1 \leq p < \infty$, the Marcinkiewicz space $\mathcal{M}^p(\Omega)$ is defined as follows:

$$\mathcal{M}^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable such that } \exists c > 0 :$$
$$\text{meas}\{x : |f(x)| \geq k\} \leq \frac{c}{k^p} \ \forall k > 0\},$$

and it is a Banach space endowed with the norm

$$\|f\|_{\mathcal{M}^p(\Omega)} = \inf \{c > 0 : \ \text{meas}\{|f| \geq k\} \leq (\frac{c}{k})^p\}.$$

Let us recall that, if Ω is bounded, for every $\epsilon \in (0, p-1]$ we have:

$$L^p(\Omega) \subseteq \mathcal{M}^p(\Omega) \subseteq L^{p-\epsilon}(\Omega)$$

with continuous embeddings.

Finally, let us explain how positive constants will be denoted hereafter. If otherwise specified, we will write simply c to denote positive constants (possibly different) which only depend on the data, that is on quantities which are fixed in the assumptions we make, as the dimension N, the bounded open set Ω, etc. Inside the proofs of our results, similar constants will also be denoted by $c_i : i = 0, 1, 2, \cdots$ to distinguish possibly different values. If we want to emphasize the dependence of one of these constants on a fixed parameter β, we will simply write c_β. In any case, the constants are always meant not to depend on the different indexes we often introduce, as n, or ϵ, which are not fixed and have a limit, for instance ϵ going to zero, or n going to infinity.

1.2. Some basic tools

We will often use the main properties of Lebesgue and Sobolev spaces which can be found, for instance, in [**Br**]. Among them, let us recall explicitly some tools which play a crucial role in the methods we use. We recall that Ω always denotes an open bounded subset of $\mathbb{R}^N$.

- *Young's inequality:* For $1 < p < \infty$, $p' = \frac{p}{p-1}$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \quad \forall a, b > 0.$$

- *Hölder's inequality:* For $1 < p < \infty$, $p' = \frac{p}{p-1}$, we have, for every f in $L^p(\Omega)$ and every g in $L^{p'}(\Omega)$

$$\left| \int_\Omega fg dx \right| \leq \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} \left(\int_\Omega |g|^{p'} dx \right)^{\frac{1}{p'}}.$$

(2.A) Let $1 < p < \infty$, and let $\{f_n\} \subset L^p(\Omega)$, $\{g_n\} \subset L^{p'}(\Omega)$ be such that f_n strongly converges to f in $L^p(\Omega)$ and g_n weakly converges to g in $L^{p'}(\Omega)$. Then

$$\lim_{n\to\infty} \int_\Omega f_n g_n dx = \int_\Omega fg dx.$$

The same conclusion holds if $p = 1$, $p' = \infty$ and the weak convergence of g_n is replaced by the weak–* convergence in $L^\infty(\Omega)$. Moreover, if f_n strongly converges to zero in $L^p(\Omega)$, and $\{g_n\}$ is bounded in $L^{p'}(\Omega)$, we also have:

$$\lim_{n\to\infty} \int_\Omega f_n g_n dx = 0.$$

(2.B) Let $\{f_n\}$ converges to f in measure and suppose that:

$$\exists c > 0,\ q > 1 : \quad \|f_n\|_{L^q(\Omega)} \leq c \quad \forall n.$$

Then

$$f_n \to f \text{ strongly in } L^s(\Omega), \text{ for every } 1 \leq s < q.$$

(2.C) *Fatou Lemma:* Let $1 \leq p < \infty$, and let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that

$$f_n \to f \text{ almost everywhere in } \Omega,$$

$$f_n \geq h(x) \text{ with } h(x) \in L^1(\Omega),$$

then

$$\int_\Omega f dx \leq \liminf_{n\to\infty} \int_\Omega f_n dx.$$

(2.D) *Generalized Lebesgue theorem:* Let $1 \leq p < \infty$, and let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that

$$f_n \to f \text{ a.e. in } \Omega$$

$$|f_n| \leq g_n \text{ with } g_n \text{ strongly converges in } L^p(\Omega).$$

Then,

$$f \in L^p(\Omega) \text{ and } f_n \text{ strongly converges to } f \text{ in } L^p(\Omega).$$

(2.E) Let $\{f_n\} \subset L^1(\Omega)$ and $f \in L^1(\Omega)$ be such that

$$f_n \geq 0, \quad f_n \to f \text{ a.e. in } \Omega,$$

$$\lim_{n\to\infty} \int_\Omega f_n dx = \int_\Omega f dx.$$

Then f_n strongly converges to f in $L^1(\Omega)$.

(2.F) *Vitali's theorem:* Let $1 \leq p < \infty$, and let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that

$$f_n \to f \text{ a.e. in } \Omega,$$

$$\lim_{\text{meas}(E)\to 0} \sup_n \int_E |f_n|^p dx = 0.$$

Then f belongs to $L^p(\Omega)$ and f_n strongly converges to f in $L^p(\Omega)$.

Note that the reverse of Vitali's theorem is also true, that is if f_n strongly converges to f in $L^p(\Omega)$, then

$$\lim_{\text{meas}(E)\to 0} \sup_n \int_E |f_n|^p dx = 0. \tag{1.2.1}$$

We will refer to this property as to the equi-integrability of the sequence $\{|f_n|^p\}$. We recall that the Dunford-Pettis theorem (see [**Br**]) says that a sequence $\{f_n\} \subset L^1(\Omega)$ is weakly convergent in $L^1(\Omega)$ if and only if it is equi-integrable. This also allows the following statement:

(2.G) Let $\{f_n\} \subset L^1(\Omega)$, $\{g_n\} \subset L^\infty(\Omega)$ be sequences such that

$$f_n \to f \text{ weakly in } L^1(\Omega),$$

$$g_n \to g \text{ weakly–* in } L^\infty(\Omega) \text{ and a.e. in } \Omega.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n g_n dx = \int_\Omega f g dx.$$

For functions in Sobolev spaces we will often use Sobolev's theorem stating that, if $p < N$, $W_0^{1,p}(\Omega)$ continuously injects into $L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$; if $p = N$, $W_0^{1,p}(\Omega)$ continuously injects into $L^q(\Omega)$ for every $q < +\infty$, while if $p > N$ then $W_0^{1,p}(\Omega)$ continuously injects into $C(\overline{\Omega})$. Let us also recall Rellich's theorem stating that, if $p < N$, the injection of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact if $1 \leq q < p^*$, and Poincare's inequality:

$$\exists\, C > 0 : \quad \|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)^N}, \quad \forall u \in W_0^{1,p}(\Omega),$$

so that $\|\nabla u\|_{L^p(\Omega)^N}$ can be used as an equivalent norm in $W_0^{1,p}(\Omega)$. Moreover, we will use several times the following result due to G. Stampacchia.

(2.H) Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(\Omega)$ we have $G(u) \in W_0^{1,p}(\Omega)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in Ω.

For a proof of *(2.H)* and related questions one can see [**KS**]. An important consequence of the previous result is that, for every $c \in \mathbb{R}$ we have

$$\nabla u = 0 \text{ a.e. in } F_c = \{x \in \Omega : u(x) = c\}. \tag{1.2.2}$$

Moreover, it allows to consider the composition of functions in $W_0^{1,p}(\Omega)$ with some useful auxiliary functions of real variable. One of the most used in what follows is the truncation function.

Definition 1.1. For $k > 0$, we define the truncation function at level $k > 0$ as

$$T_k(s) = \max(-k, \min(k, s)) = \begin{cases} s & \text{if } |s| \leq k \\ k & \text{if } s > k \\ -k & \text{if } s < -k; \end{cases}$$

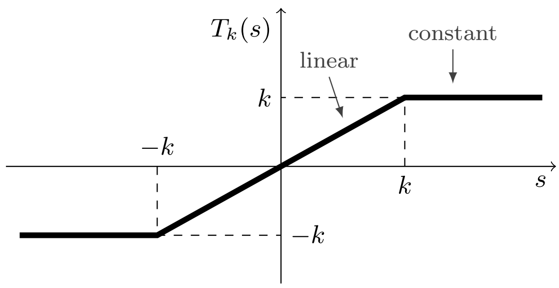


FIGURE 1. The function $T_k(s)$

$$G_k(s) = s - T_k(s) = (|s| - k)_+ \text{sign}(s)$$

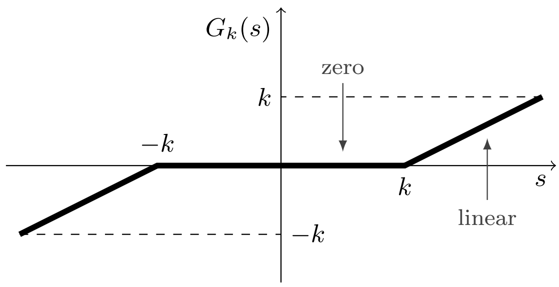


FIGURE 2. The function $G_k(s)$

Then if u belongs to $W_0^{1,p}(\Omega)$, it follows that $T_k(u)$ also belongs to $W_0^{1,p}(\Omega)$ and

$$\nabla T_k(u) = \nabla u \chi_{\{|u|\le k\}}, \quad \nabla G_k(u) = \nabla u \chi_{\{|u|\ge k\}} \text{ a.e. on } \Omega, \text{ for every } k > 0. \tag{1.2.3}$$

If u is such that its truncation belongs to $W_0^{1,p}(\Omega)$, then we can define an *approximated gradient* of u defined as the a.e. unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $v = \nabla T_k(u)$ almost everywhere on the set $\{|u| \le k\}$, for every $k > 0$ (see [**B6**]).

1.3. Elliptic operators on classical Sobolev spaces

Let us recall that a function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is called a Carathéodory function if the function $x \mapsto a(x, s, \zeta)$ is measurable for every (s, ζ) in $\mathbb{R} \times \mathbb{R}^N$ and $(s, \zeta) \mapsto a(x, s, \zeta)$ is continuous for almost every x in Ω. If $u : \Omega \to \mathbb{R}$, $v : \Omega \to \mathbb{R}^N$ are measurable functions, then $a(x, u(x), v(x))$ is measurable in Ω, so that Carathéodory functions are used to define composition operators on Lebesgue or Sobolev spaces (see [**Vai**]).

We will say that a Carathéodory function $a(x, s, \zeta)$ satisfies the Leray-Lions assumptions (see [**LL**]) if there exists $p > 1$ such that, for almost every x in Ω, for every s in $\mathbb{R}$ and every ζ, η in $\mathbb{R}^N$:

$$(a1) \qquad a(x, s, \zeta) \cdot \zeta \ge \alpha_0 |\zeta|^p, \quad \alpha_0 > 0,$$

$$(a_2) \qquad |a(x, s, \zeta)| \le \beta(a_2(x) + |s|^{p-1} + |\zeta|^{p-1}) \quad \beta > 0, \ a_2(x) \in L^{p'}(\Omega),$$

$$(a_3) \qquad (a(x, s, \zeta) - a(x, s, \eta)) \cdot (\zeta - \eta) > 0 \text{ for evry } \zeta \ne \eta.$$

REMARK 1.2. It should be noted that assumption $(a1)$ implies that $a(x, s, 0) = 0$ for every s in $\mathbb{R}$. This follows from the fact $a(x, s, t\zeta) > 0$ if $t > 0$ and $a(x, s, t\zeta) < 0$ if $t < 0$, and since $a(x, s, \zeta)$ is Carathéodory (hence $\zeta \mapsto a(x, s, \zeta)$ is continuous).

Note that $(a1) - (a3)$ imply that the divergence form operator $A(u) = -\text{div}(a(x, u, \nabla u))$ is well defined, bounded from the Sobolev space $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ and has coercivity and monotonicity properties. The main result proved by J. Leray and J.-L. Lions is that A is surjective on $W^{-1,p'}(\Omega)$. Let us recall this result

THEOREM 1.3. *Let $a(x,s,\zeta)$ be a bounded Carathéodory function and let f belongs to $W^{-1,p'}(\Omega)$. Then there exists $u \in W_0^{1,p}(\Omega)$ which is a weak solution of*

$$\begin{cases} -\operatorname{div}(a(x,s,\nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that u satisfies

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

PROOF. See [**L, LL**]. □

The proof of this theorem, which use Schauder's fixed point theorem, relies on a compactness argument where the strict monotonicity assumption $(a3)$ plays a crucial role. A basic ingredient in this method is a lemma which we present here, in a slightly modified version, since it will be very often used in the sequel.

LEMMA 1.4. *Let $a(x,s,\zeta)$ satisfy $(a1)-(a3)$ and let $\{v_n\}$, $\{w_n\}$ be such that:*

$$v_n \to v \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega,$$

$$w_n \to w \text{ weakly in } L^p(\Omega)^N.$$

Assume that

$$\lim_{n\to+\infty}\int_\Omega (a(x,v_n,w_n) - a(x,v_n,w))(w_n - w)dx = 0.$$

Then we have, up to a subsequence,

$$w_n \to w \text{ strongly in } L^p(\Omega)^N \text{ and a.e. in } \Omega.$$

PROOF. See [**BMP**], Lemma 5. □

A whole theory has recently developed about the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.1}$$

where $a(x,s,\zeta)$ satisfies $(a1)-(a3)$ and μ belongs to $\mathcal{M}_b(\Omega)$, the space of bounded Radon measures on Ω. The interest is studying problem (1.3.1) arises if $p \le N$, since if $p > N$ then $\mathcal{M}_b(\Omega) \subset W^{-1,p'}(\Omega)$ by Sobolev embedding theorem and to (1.3.1) it can be applied Theorem 1.3. On the other hand, if $p \le N$, we cannot expect to have solutions of (1.3.1) in $W_0^{1,p}(\Omega)$, nor it is clear in which sense the equation should be considered. In the linear case, i.e. if $a(x,s,\zeta) = A(x)\zeta$, with $A(x)$ a bounded and coercive matrix, problem (1.3.1) has been exhaustively studied in [**S**] using a duality argument. In the general nonlinear case, the key point in finding solutions of (1.3.1) is the following standard approximation result in $\mathcal{M}_b(\Omega)$. Henceforward, we will say that a sequence $\{\mu_n\} \subset \mathcal{M}_b(\Omega)$ converges tightly to a measure μ if

$$\lim_{n\to\infty}\int_\Omega \varphi d\mu_n = \int_\Omega \varphi d\mu \quad \forall\varphi \in C(\overline{\Omega}). \tag{1.3.2}$$

Let us remark that μ_n tightly converges to μ if and only if μ_n converges to μ in the weak–* topology of $\mathcal{M}_b(\Omega)$ and $\mu_n(\Omega)$ converges to $\mu(\Omega)$.

THEOREM 1.5. *Let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a sequence $\{f_n\} \subset C^\infty(\Omega)$ such that*

$$\|f_n\|_{L^1(\Omega)} \le \|\mu\|_{\mathcal{M}_b(\Omega)},$$

$$f_n \to \mu \text{ tightly in } \mathcal{M}_b(\Omega).$$

Thanks to Theorem 1.5, a method for solving (1.3.1) is to find *a priori estimates* which only depend on the L^1-norm of the datum f and then look for compactness results which allow to pass to the limit in approximating problems. This method has been proved to work in **[BG1]** and yields a function u which is a distributional solution of (1.3.1). However, u only belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$, and this regularity is optimal as showed by simple examples (for instance, the fundamental solution of the $p-$laplacian equation in a Ball of $\mathbb{R}^N$). Since $\frac{N(p-1)}{N-1} > 1$ if and only if $p > 2 - \frac{1}{N}$, for smaller values of p we cannot even use the framwork of Sobolev spaces to deal with (1.3.1), so that this lower bound on p is required in **[BG1]**. This obstacle has been overcome in **[B6]** by using the properties enjoyed not by u but by its truncations $T_k(u)$, for which *a priori estimates* in the space $W_0^{1,p}(\Omega)$ are always available. Let us then precise this new functional setting and recall some of the known results.

Definition 1.6. We define $\mathcal{T}_0^{1,p}(\Omega)$ as the set of measurable functions $u : \Omega \to \mathbb{R}$ almost everywhere finite and such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$.

With very easy examples it can be checked that $\mathcal{T}_0^{1,p}(\Omega)$ is not even a vector space. However, if u is in $\mathcal{T}_0^{1,p}(\Omega)$ and φ is in $W_0^{1,p}(\Omega) \cap L^\infty(Q)$ then $u + \varphi$ belongs to $\mathcal{T}_0^{1,p}(\Omega)$. The importance of the space $\mathcal{T}_0^{1,p}(\Omega)$ is that it is possible to extend the notion of gradient to this class of functions.

Lemma 1.7. *Let u belong to $\mathcal{T}_0^{1,p}(\Omega)$. Then there exists a unique measurable function $\nabla u : \Omega \to \mathbb{R}^N$, such that*

$$\nabla T_k(u) = \nabla u \chi_{\{|u|<k\}} \text{ a.e. in } \Omega \quad \forall k > 0.$$

Moreover u belongs to $W_0^{1,1}(\Omega)$ if and only if ∇u, as defined above, belongs to $L^1(\Omega)^N$, and in this case it coincides with the usual notion of gradient in Sobolev spaces.

Proof. See **[B6]**, Lemma 2.1. □

We can now provide the definition of weak solution for (1.3.1), and the gradient appearing in the equation will henceforth be the gradient as defined in Lemma 1.7.

Definition 1.8. A function u in $\mathcal{T}_0^{1,p}(\Omega)$ is a weak solution of (1.3.1) if $a(x, u, \nabla u)$ belongs to $L^1(\Omega)^N$ and the equation is satisfied in the sense of distributions, that is

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla dx = \int_\Omega \varphi d\mu, \quad \forall \varphi \in C_c^\infty(\Omega).$$

In **[BG1]**, for $p > 2 - \frac{1}{N}$, and in **[B6]** (see also **[BGO1]**) in the general case, the problem of existence of weak solutions of (1.3.1) is solved by using the following tools, which we here recall for further purposes.

Lemma 1.9. *Let $C > 0$ and let $\{u_n\} \subset \mathcal{T}_0^{1,p}(\Omega)$ be such that:*

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le C(k+1) \quad \forall k > 0.$$

Then if $p < N$, $\{u_n\}$ is bounded in $M^{\frac{N(p-1)}{N-p}}$ and $\{|\nabla u_n|\}$ is bounded in $M^{\frac{N(p-1)}{N-1}}(\Omega)$; and if $p = N$, $\{u_n\}$ is bounded in $M^q(\Omega)$ for every $q < +\infty$ and $\{|\nabla u_n|\}$ is bounded in $M^r(\Omega)$ for every $r < N$. Moreover, there exist a measurable function u in $\mathcal{T}_0^{1,p}(\Omega)$ and a subsequence, not relabeled, such that

$$u_n \to u \text{ a.e. in } \Omega,$$

$$T_k(u_n) \to T_k(u) \text{ weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega \text{ for every } k > 0.$$

Proof. As far as the estimates are concerned, see **[B6]**, Lemma 4.1 and Lemma 4.2 if $p < N$, while for the case $p = N$ see **[BPV]**, Lemma 2.5. The convergence results are contained in Theorem 6.1 of **[B6]**. □

Proposition 1.10. *Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be solution of*

$$\begin{cases} -\operatorname{div}(a(x, u_n, \nabla u_n)) = f_n - \operatorname{div}(F_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\{f_n\} \subset L^\infty(\Omega)$ *are such that* $\|f_n\|_{L^1(\Omega)} \le C$, *and* $\{F_n\} \subset L^\infty(\Omega)^N$ *strongly converges in* $L^{p'}(\Omega)^N$. *Then there exist* u *in* $\mathcal{T}_0^{1,p}(\Omega)$, *and a subsequence, not relabeled, such that*

$$u_n \to u \text{ a.e. in } \Omega,$$

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega,$$

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \text{ strongly in } L^1(\Omega)^N.$$

PROOF. See [**B6, BGO1, BPV**]. □

Thanks to Proposition 1.10 and to Lemma 1.9 it follows the existence result for (1.3.1).

THEOREM 1.11. *Assume* $(a1)-(a3)$, *and let* μ *belong to* $\mathcal{M}_b(\Omega)$. *Then there exists a weak solution* u *of* (1.3.1) *in* $\mathcal{T}_0^{1,p}(\Omega)$. *Moreover if* $p < N$, u *belongs to* $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$ *and* $|\nabla u|$ *belongs to* $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$, *while if* $p = N$, u *belongs to* $\mathcal{M}^q(\Omega)$ *for every* $q < +\infty$ *and* $|\nabla u|$ *belongs to* $\mathcal{M}^r(\Omega)$ *for every* $r < N$.

PROOF. See [**B6**], Theorem 6.1 for $p < N$, and [**BPV**], Theorem 2.6 for $p = N$. □

If $1 < p < N$ and μ is a function belonging to $L^r(\Omega)$, with $1 < r < (p^*)'$, the function u which is given by Theorem 1.11 can be proved to be more regular.

PROPOSITION 1.12. *Let* $1 < p < N$, *and let* μ *belong to* $L^r(\Omega)$, $1 < r < (p^*)'$. *Then there exists a weak solution* u *of* (1.3.1) *such that* $|\nabla u|^{p-1}r^*$ *and* $|u|^{((p-1)r^*)^*}$ *belong to* $L^1(\Omega)$.

PROOF. See [**BG2**]. □

1.4. Elliptic capacity and Measures

Nothing has been said until now on the problem of uniqueness of solutions of (1.3.1), we will not be concerned with uniqueness problems, nevertheless let us just recall that a counterexample by J. Serrin [**Ser**] shows that uniqueness may fail even for linear operators in the class of distributional solutions belonging to $W^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ (in this case $p = 2$). Since this example is constructed with a distributional solution which is not proved to be in $\mathcal{M}_b(\Omega)$, uniqueness of weak solutions as defined in Definition 1.8 is still an open problem. The attempt to find a different formulation of (1.3.1) which could allow to have both existence and uniqueness has been developed in [**B6**] and in [**LM**] where the notions of entropy solution and renormalized solution have been respectively introduced. Both these definitions (which have been proved to be equivalent, see for instance [**DMOP**]) ask for solutions in $\mathcal{M}_b(\Omega)$ and use a weak formulation of the equation where nonlinear test functions depending on u are used to restrict the equation on the subsets where u is bounded. Both these approaches are able to get uniqueness provided μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. In terms of measures, this restriction has a straight relationship with the notion of $p-$capacity, as it was proved in [**BGO1**]. In order to recall this result, we need first to introduce the notion of capacity (See [**DMOP**], Section 2 for details).

For $p > 1$, the $p-$capacity of a compact set K of Ω can be defined as follows (χ_K denotes the characteristic function of K):

$$\text{cap}_p(K) = \inf\left\{\int_\Omega |\nabla u|^p dx,\ u \in C_c^\infty(\Omega,\ u \ge \chi_K)\right\},$$

with the convention that $\text{cap}_p(\emptyset) = +\infty$. This definition can then be extended first to open sets A, then to every borelian subset B of Ω, by setting:

$$\text{cap}_p(A) = \sup\{\text{cap}_p(K),\ K \subset A,\ K \text{ compact}\},$$

$$\text{cap}_p(B) = \sup\{\text{cap}_p(A),\ B \subset A,\ A \text{ open}\}.$$

Let us also recall that a function u is said to be cap_p quasi-continuous if for every $\epsilon > 0$ there exists a set $E \subset \Omega$ such that $\text{cap}_p < \epsilon$ and u is continuous in $\Omega\backslash E$. It is well known that every function u in $W_0^{1,p}(\Omega)$ admits a unique cap_p quasi-continuous representative $\tilde{u}$ in $W_0^{1,p}(\Omega)$, that is a function $\tilde{u}$ which is equal to u almost everywhere in Ω and is cap_p quasi-continuous. Moreover the values of $\tilde{u}$ are defined cap_p quasi-everywhere. Thanks to this fact it is also possible to prove that

(2.I) For u in $W_0^{1,p}(\Omega)$, letting $\tilde{u}$ be the cap_p quasi-continuous representative of u, for every Borel set $B \subset \Omega$ we have

$$\mathrm{cap}_p(B) = \inf\left\{ \int_\Omega |\nabla u|^p dx, u \in W_0^{1,p}(\Omega),\ \tilde{u} \geq \chi_B \text{ quasi-everywhere in } \Omega \right\},$$

(2.J) If u belongs to $W_0^{1,p}(\Omega)$, and μ is a bounded Radon measure such that $\mu(E) = 0$ for every $E \subset \Omega$ such that $\mathrm{cap}_p(E) = 0$, we have that u is measurable with respect to μ and, if u is also bounded, then u belongs to $L^\infty(\Omega, d\mu)$ (see also **[DMOP]**, Proposition 2.7).

Then, if a function u belongs to $\mathcal{T}_0^{1,p}(\Omega)$, its truncation $T_k(u)$ has a cap_p quasi-continuous representative $\tilde{u}_k$, a natural question is whether u itself may admit a cap_p quasi-continuous representative $\tilde{u}$. Simple examples show that in general this is false without further assumptions on u, however it can be proved to be true if u also satisfies the estimate:

$$\|T_k(u)\|^p_{W_0^{1,p}(\Omega)} \leq C(k+1) \quad \forall k > 0,$$

which we know to hold true for solutions of elliptic equations with measure data. Let us give a proof of this fact, which is established in **[DMOP]**.

LEMMA 1.13. *Let u be in $\mathcal{T}_0^{1,p}(\Omega)$ and assume that there exists $C > 0$ such that:*

$$\|T_k(u)\|^p_{W_0^{1,p}(\Omega)} \leq C(k+1) \quad \forall k > 0.$$

Then u is cap_p quasi-continuous finite (i.e. $\mathrm{cap}_p(\{x : |u(x)| = +\infty\}) = 0$) and there exists a cap_p quasi-continuous representative $\tilde{u}$ of u (i.e. $u = \tilde{u}$ almost everywhere in Ω and $\tilde{u}$ is cap_p quasi-continuous).

PROOF. Let us call $\tilde{u}_k$ the cap_p quasi-continuous representative of $T_k(u)$ in $W_0^{1,p}(\Omega)$. We define

$$\tilde{u} = \tilde{u}_k \text{ in } \{x : |u(x)| < k\}.$$

Let us first observe that if $k > j$ then $\{x : |u(x)| < j\} \subset \{x : |u(x)| < k\}$ and $T_k(u) = T_j(u)$ almost everywhere in $\{x : |u(x)| < j\}$, so that:

$$\tilde{u}_k = \tilde{u}_j \text{ a.e. in } \{x : |u(x)| < j\}.$$

Thus $\tilde{u}$ is well defined (almost everywhere) in Ω, and

$$u = T_k(u) = \tilde{u}_k = \tilde{u} \text{ a.e. in } \{x : |u(x)| < k, \text{ for any } k > 0\},$$

hence $u = \tilde{u}$ almost everywhere in Ω. Moreover, thanks to *(2.I)*, it is possible to use $\frac{T_k(u)}{k}$ as test function for the $p-$capacity of the set $\{x : |T_k(u(x))| \geq k\}$, so that we have:

$$\mathrm{cap}_p(\{x : |u(x)| \geq k\}) = \mathrm{cap}_p(\{x : |T_k(u(x))| \geq k\}) \leq \frac{1}{k^p}\|T_k(u)\|^p_{W_0^{1,p}(\Omega)} \leq C\frac{k+1}{k^p}, \tag{1.4.1}$$

so that letting k tend to infinity we deduce that u is cap_p quasi-everywhere finite. Moreover, given $\epsilon > 0$, we can fix k_ϵ such that:

$$\mathrm{cap}_p(\{x : |u(x)| \geq k_\epsilon\}) \leq \epsilon.$$

Since $\tilde{u}_{k_\epsilon}$ is a cap_p quasi-continuous function, there exists $F_\epsilon \subset \Omega$ such that $\mathrm{cap}_p(F_\epsilon) \leq \epsilon$ and $\tilde{u}_{k_\epsilon}$ is continuous in $\Omega \backslash F_\epsilon$. Let now $E = F_\epsilon \cup \{x : |u(x)| \geq k_\epsilon\}$. Then $\mathrm{cap}_p(E) \leq 2\epsilon$ and in $\Omega\backslash E$ we have $\tilde{u} = \tilde{u}_{k_\epsilon}$ which is continuous. This proves that $\tilde{u}$ is cap_p quasi-continuous. □

We define $\mathcal{M}_0(\Omega)$ as the set of all measures μ in $\mathcal{M}_b(\Omega)$ which are "*absolutely continuous*" with respect to the $p-$capacity, i.e., which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\mathrm{cap}_p(B, \Omega) = 0$. We define $\mathcal{M}_s(\Omega)$ as the set of all measures μ in $\mathcal{M}_b(\Omega)$ which are "*singular*" with respect to the $p-$capacity, i.e., the measures for which there exists a Borel set $E \subset \Omega$, with $\mathrm{cap}_p(E, \Omega) = 0$, such that $\mu \perp E$. The following result is the analogue of the Lebesgue decomposition theorem, and can be proved in the same way.

PROPOSITION 1.14. *For every measure μ in $\mathcal{M}_b(\Omega)$, there exists a unique pair of measures (μ_0, μ_s), with μ_0 in $\mathcal{M}_0(\Omega)$ and μ_s in $\mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. If μ is nonnegative, so are μ_0 and μ_s.*

PROOF. See **[FST]**, Lemma 2.1. □

The measures μ_0 and μ_s will be called the *absolutely continuous* and the *singular* part of μ with respect to the $p-$capacity. To deal with μ_0 we need a further decomposition result.

PROPOSITION 1.15. *Let μ_0 be a measure in $\mathcal{M}_b(\Omega)$. Then μ_0 belongs to $\mathcal{M}_0(\Omega)$ if and only if it belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Thus, if μ_0 belongs to $\mathcal{M}_0(\Omega)$, there exist $f \in L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$, such that*

$$\mu_0 = f - \operatorname{div}(g), \tag{1.4.2}$$

in the sense of distributions; moreover one has

$$\int_\Omega v d\mu_0 = \int_\Omega f v dx + \int_\Omega g \cdot \nabla v \; dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Note that the decomposition (1.4.2) is not unique since $L^1(\Omega) \cap W^{-1,p'}(\Omega) \neq \{0\}$.

PROOF. See [**BGO1**], Theorem 2.1. □

Putting together the results of Propositions 1.14 and 1.15, and the Hahn decomposition theorem, we obtain the following result.

PROPOSITION 1.16. *Every measure μ in $\mathcal{M}_b(\Omega)$ can be decomposed as follows*

$$\mu = \mu_0 + \mu_s = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-, \tag{1.4.3}$$

where μ_0 is a measure in $\mathcal{M}_0(\Omega)$, and so it can be written as $f - \operatorname{div}(g)$, with f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$, while μ_s^+ and μ_s^- (the positive and negative parts of μ_s) are two nonnegative measures in $\mathcal{M}_b(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero $p-$capacity. We set $E = E^+ \cup E^-$.

The following technical propositions will be used several times in what follows; the second one is a well-known consequence of the Egorov's theorem.

PROPOSITION 1.17. *Let μ_0 be a measure in $\mathcal{M}_0(\Omega)$, and let v be a function in $W_0^{1,p}(\Omega)$. Then (the cap_p quasi continuous representative of) v is measurable with respect to μ_0. If v further belong to $L^\infty(\Omega)$, then (the cap_p quasi continuous representative of) v belongs to $L^\infty(\Omega, \mu_0)$, hence to $L^1(\Omega, \mu_0)$.*

PROOF. Every cap_p quasi-continuous function coincides cap_p quasi-everywhere with a Borel function and therefore measurable for any measure μ_0 in $\mathcal{M}_0(\Omega)$, since these measures do not charge sets of zero $p-$capacity. If v belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then there exist a constant k such that $|v| \leq k$ almost everywhere on Ω. Consequently the cap_p quasi-continuous representative of v satisfies $|v| \leq k$ cap_p quasi-everywhere on Ω (see [**HKM**], Theorem 4.12), and thus μ_0-almost everywhere on Ω. □

PROPOSITION 1.18. *Let Ω be a bounded, open subset of $\mathbb{R}^N$, ρ_ϵ be a sequence of $L^1(\Omega)$ functions that converges to ρ weakly in $L^1(\Omega)$, and let σ_ϵ be a sequence of functions in $L^\infty(\Omega)$ that is bounded in $L^\infty(\Omega)$ and converges to σ almost everywhere in Ω. Then*

$$\lim_{\epsilon \to 0} \int_\Omega \rho_\epsilon \sigma_\epsilon dx = \int_\Omega \rho\sigma dx.$$

let us recall some fundamental results on the link between $p-$capacity and Radon measures.

THEOREM 1.19. *Let μ belong to $\mathcal{M}_b(\Omega)$. Then $\mu(E) = 0$ for every subset $E \subset \Omega$ such that $cap_p(E) = 0$ if and only if μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.*

PROOF. See [**BGO1**], Theorem 2.1. □

THEOREM 1.20. *Let μ belong to $\mathcal{M}_b(\Omega)$. Then there exist a unique couple of measures (μ_0, λ) such that $\mu_0, \lambda \in \mathcal{M}_b(\Omega)$, $\mu_0(B) = 0$ for every subset B such that $cap_p(B) = 0$ while λ is concentrated (see 1.1.1) on a subset E of zero $p-$capacity, and $\mu = \mu_0 + \lambda$. By Theorem 1.19 we then have that there exist $f \in L^1(\Omega)$, F in $L^{p'}(\Omega)^N$, such that:*

$$\mu = f - \operatorname{div}(F) + \lambda.$$

Moreover, if $\mu \geq 0$, we have $\mu_0 \geq 0$, $\lambda \geq 0$ and also f can be chosen positive.

PROOF. See [**FST**], Lemma 2.1. □

Let us remark that, since $L^1(\Omega)\cap W^{-1,p'}(\Omega)\neq\{0\}$, there is not a unique way, in the above Theorem 1.20, to write $\mu_0 = f - \operatorname{div}(F)$, with f in $L^1(\Omega)$ and $F\in L^{p'}(\Omega)^N$.

In virtue of Theorems 1.19 and 1.20, the measure μ can be splitted as follows

$$\mu=\mu_0+\lambda, \quad \mu_0 = f-\operatorname{div}(F), \tag{1.4.4}$$

where μ_0, $\lambda\in\mathcal{M}_b(\Omega)$, f is in $L^1(\Omega)$, F is in $(L^{p'}(\Omega))^N$ and $\lambda=\lambda\perp E$, that is λ is concentrated on a set E such that $\operatorname{cap}_p(E)=0$. In what follows we will always make use of the previous decomposition of μ, and moreover in the case $\mu\in\mathcal{M}_b^+(\Omega)$, that is μ is a positive measure, we also have, in (1.4.4), that $f\geq 0$, μ_0 and $\lambda\geq 0$. Then we consider an approximation (μ_n) of μ, for instance, it can be obtained by convolution of μ with mollifying Kernel, satisfying the following conditions:

$$\begin{cases}\mu_n = f_n - \text{ div } (f_n)+\lambda_n,\\ \mu_n\in C^\infty(\Omega),\ \exists C>0:\ \|\mu_n\|_{L^1(\Omega)}\leq C\ \forall n,\\ f_n\to f \text{ weakly in } L^1(\Omega),\\ F_n\to F \text{ strongly in } L^{p'}(\Omega)^N,\\ \lambda_n\to\lambda \text{ tightly, i.e. } \int_\Omega\varphi d\lambda_n\to\int_\Omega\varphi d\lambda \quad \forall\varphi\in C(\overline{\Omega}).\end{cases} \tag{1.4.5}$$

Then, there exist solutions u_n in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ of the approximating Dirichlet problems:

$$\begin{cases}-\operatorname{div}(a(x,u_n,\nabla u_n))=\mu_n & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega,\end{cases} \tag{1.4.6}$$

The key point in our general study is contained in the following compactness result on the sequence of truncations $\{T_k(u_n)\}$, proved in **[DMOP]**. Two ingredients will be essential, the first one is contained in the following lemma.

LEMMA 1.21. *Let $\delta>0$. Then there exists a compact set $K_\delta\subset E$ and there exists a sequence (ψ_δ) of functions in $C_c^\infty(\Omega)$ such that:*

$$\begin{gathered}\psi_\delta\in C_c^\infty(\Omega), \quad 0\leq\psi_\delta\leq 1, \quad \psi_\delta\equiv 1 \text{ on } K_\delta, \quad \lambda(E\backslash K_\delta)<\delta\\ \psi_\delta\to 0 \text{ strongly in } W_0^{1,p}(\Omega) \text{ as } \delta \text{ tends to zero.}\end{gathered} \tag{1.4.7}$$

PROOF. The existence of a compact set K_δ such that $\lambda(E\backslash K_\delta)<\delta$ follows from the fact that λ belongs to $\mathcal{M}_b(\Omega)$, so that it is a regular Borel measure. Since we have that $\operatorname{cap}_p(K_\delta)=0$, and since K_δ is compact, the existence and the properties of ψ_δ follows from the definition of $p-$capacity. □

THEOREM 1.22. *Let $\mu\in\mathcal{M}_b(\Omega)$ and $1\leq p<N$. Let μ_n be an approximation of μ in the sense of* (1.4.5). *Assume that* $(a1)-(a3)$ *hold true, and let u_n be solution of* (1.4.6). *Then there exist a measurable function u in $\mathcal{T}_0^{1,p}(\Omega)$, and a subsequence such that*

$$\begin{cases}T_k(u_n)\to T_k(u) & \text{strongly in } W_0^{1,p}(\Omega) \text{ for every } k>0,\\ \nabla u_n\to\nabla u & \text{a.e. in } \Omega,\\ a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) & \text{strongly in } (L^q(\Omega))^N \text{ for every } 1<q<\frac{N}{N-1}.\end{cases} \tag{1.4.8}$$

PROOF. See **[DMOP, Ma]**. □

1.5. Duality solutions

Let $A:\Omega\to\mathbb{R}^{N^2}$ be a matrix-valued measurable function such that there exist $0<\alpha\leq\beta$ such that

$$A(x)\ \zeta.\zeta\geq\alpha|\zeta|^2, \quad |A(x)|\leq\beta, \tag{1.5.1}$$

for almost every x in Ω, and for every ζ in $\mathbb{R}^N$. Consider the following uniformly elliptic equations with Dirichlet boundary conditions

$$\begin{cases}-\operatorname{div}(A(x)\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.5.2}$$

where f is a function defined on Ω which satisfies suitable assumptions. If the matrix A is the identity matrix, problem (1.5.2) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., the Dirichlet problem for the Laplacian operator. Now consider the $N \times N$ matrix $A(x)$ with entries $a_{i,j}(x) \in L^\infty(\Omega)$ satisfying assumption (1.5.1) ($p = 2$), and consider u and v be the solutions of the linear problems

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\mathrm{div}(A^*(x)\nabla v) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5.3}$$

where A^* is the transposed matrix of A (note that A^* satisfies (1.5.1) with the same constants as A). If $f \in W^{-1,p'}(\Omega)$, with $p' > N$ we can consider

$$\begin{cases} -\mathrm{div}(A^*(x)\nabla v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.4}$$

Let v be the variation solution of problem (1.5.4); thanks to standard elliptic regularity results we have that $v \in C(\overline{\Omega})$ and

$$\|v\|_{C(\overline{\Omega})} \leq \lambda \|f\|_{W^{-1,p'}(\Omega)}. \tag{1.5.5}$$

So, for every $p' > N$, we can define $G^*_{p'} : W^{-1,p'}(\Omega) \longrightarrow C(\overline{\Omega})$, as $G^*_{p'}(f) = v$, $G^*_{p'}$ turns out to be linear and continuous; thus we can define the *Green operator* as

$$G^* : \bigcup_{p'>N} W^{-1,p'}(\Omega) \longrightarrow C_0(\Omega),$$

with $G^* |_{W^{-1,p'}(\Omega)} = G^*_{p'}$. This argument justifies the definition of *duality solution* given by G. Stampacchia in **[S]**, for the problem

$$\begin{cases} -\mathrm{div}(A(x)u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.6}$$

DEFINITION 1.23. Let $\mu \in \mathcal{M}_b(\Omega)$, we will say that $u \in L^1(\Omega)$ is a *duality solution* of problem (1.5.6) if

$$\int_\Omega ug\, dx = \int_\Omega G^*(g)\, d\mu \quad \text{for all } g \in L^\infty(\Omega). \tag{1.5.7}$$

A duality solution, easily, turns out to be a distributional solution of problem (1.5.6) and, if it exists, is obviously unique as an easy consequence of its definition.

THEOREM 1.24. *Let $\mu \in \mathcal{M}_b(\Omega)$, then there exists a unique duality solution of problem (1.5.6). Moreover, $u \in W^{1,q}_0(\Omega)$ with $q < \frac{N}{N-1}$.*

PROOF. See **[S]**. □

REMARK 1.25. Notice that the regularity of the duality solution, that is $u \in W^{1,q}_0(\Omega)$ with $q < \frac{N}{N-1}$, is sharp and cannot be, in general, improved, in fact one can think about the fundamental solution of the Laplace operator in a ball. So, in general, we deal with solutions that do not belong to the usual energy space. However notice that, as we will see below, these *infinity energy solutions* turn out to have finite energy truncations at any level.

1.6. Non–uniqueness for distributional solutions

If the datum μ is a measure, we have that the sequence u_n of approximating solutions is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. Therefore, and up to subsequences, u_n weakly converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Choosing a test function $\varphi \in C_0^1(\Omega)$ in the weak formulation (1.5.6) for (u_n, μ_n), we obtain

$$\int_\Omega A(x)\nabla u_n \cdot \nabla\varphi \, dx = \int_\Omega f_n\varphi \, dx - \int_\Omega F_n \cdot \nabla\varphi \, dx, \tag{1.6.1}$$

which, passing to the limit, yields

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi \, d\mu \quad \forall\varphi \in C_0^1(\Omega),$$

so that u is a solution in *the sense of distributions*. Since the definition of solution in the sense of distributions can always be given (even when the notion of duality solution is unavailable due for example to the operator being nonlinear), one may wonder whether there is a way of proving uniqueness of distributional solutions (not passing through duality solutions). The following example is due to J. Serrin [**Ser**].

EXAMPLE. Let $\epsilon > 0$ and $A^\epsilon(x)$ be the symmetric matrix defined by

$$a_{ij}^\epsilon(x) = \delta_{ij} + (a_\epsilon - 1)\frac{x_i x_j}{|x|^2}. \tag{1.6.2}$$

If $a_\epsilon = \frac{N-1}{\epsilon(N-2+\epsilon)}$, then the function $\omega^\epsilon(x) = x_1|x|^{1-N-\epsilon}$ is a solution in the sense of distributions of

$$-\mathrm{div}(A^\epsilon(x)\nabla\omega^\epsilon) = 0, \quad \mathbb{R}^N\backslash\{0\}. \tag{1.6.3}$$

Indeed, if we rewrite $\omega(x) = x_1|x|^\alpha$ and $a_{ij}(x) = \delta_{ij} + \beta\frac{x_i x_j}{|x|^2}$. Simple calculations imply

$$\omega_{x_1}(x) = |x|^\alpha + \alpha x_1^2|x|^{\alpha-2}, \quad \omega_{x_i} = \alpha x_1 x_i|x|^{\alpha-2},$$

so that $\mathrm{div}(A(x)\nabla\omega) = x_1|x|^{\alpha-2}[\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta)]$.
Given $0 < \epsilon < 1$, if we choose $\alpha = 1-N-\epsilon$, and $\beta = \frac{N-1}{\epsilon(N-1+\epsilon)}$, we have

$$\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta) = 0,$$

so that ω is a solution of (1.6.3) if $x \neq 0$. Let now $\Omega = B_1(0)$ be the unit ball, and v_ϵ be the unique solution of

$$\begin{cases} -\mathrm{div}(A^\epsilon(x)\nabla v^\epsilon) = \mathrm{div}(A^\epsilon(x)\nabla x_1) & \text{in } \Omega, \\ v^\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists since $\mathrm{div}(A^\epsilon\nabla x_1)$ is a regular function belonging to $H^{-1}(\Omega)$. Therefore, the function $z^\epsilon = v^\epsilon + x_1$ is the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A^\epsilon(x)\nabla u^\epsilon) = 0 & \text{in } \Omega, \\ u^\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which is not identically zero since z^ϵ belongs to $H^1(\Omega)$, while w^ϵ belongs to $W_0^{1,q}(\Omega)$ for every $q < q_\epsilon = \frac{N}{N-1+\epsilon}$.
Hence, the problem

$$\begin{cases} -\mathrm{div}(A^\epsilon(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has infinitely many solutions in the sense of distributions, which can be written as $u = \overline{u} + tu^\epsilon$, t in $\mathbb{R}$, where $\overline{u}$ is the duality solution.

One may observe that the solution found by approximation belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, while the solution of the above example belongs to $W_0^{1,q}(\Omega)$ for some $q < \frac{N}{N-1}$, and that we are not allowed to take $\epsilon = 0$ since in this case a_ϵ diverges. Thus one may hope that there is still uniqueness of the solution obtained by approximation. However it is possible to modify Serrin's example in dimension $N \geq 3$ (see [**Pr1**]) to find a non-zero solution in the sense of distributions for

$$\begin{cases} -\mathrm{div}(B^\epsilon(x)\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Here

$$B^\epsilon(x) = \begin{pmatrix} 1 + (a_\epsilon - 1)\frac{x_1^2}{x_1^2+x_2^2} & (a_\epsilon - 1)\frac{x_1 x_2}{x_1^2+x_2^2} & 0 \\ (a_\epsilon - 1)\frac{x_1 x_2}{x_1^2+x_2^2} & 1 + (a_\epsilon - 1)\frac{x_1^2}{x_1^2+x_2^2} & 0 \\ 0 & 0 & I \end{pmatrix}$$

where I is the identity matrix in $\mathbb{R}^{N-2}$, and a_ϵ is as above, with ϵ fixed so that $\omega^\epsilon(x) = x_1(\sqrt{x_1^2 + x_2^2})^{\epsilon-1}$ belongs to $W^{1,q}(\mathbb{R}^2)$ for every $q < 2$. On the other hand, in dimension $N = 2$ there is a unique solution in the sense of distributions belonging to $W_0^{1,q}(\Omega)$, for every $q < 2$. The proof of this fact uses Meyer's regularity theorem for linear equations with regular data.

1.7. Entropy solutions

As we have seen, uniqueness of solutions for distributional solutions can fail even in the linear case if the regularity of the solutions is not "enough" to allow the choice of less regular test functions. And the lack of regularity of the solution of the counter-example by Serrin (as modified in [**Pr1**]) is exactly the one which is typical of the solutions of equations with data in $L^1(\Omega)$ or in $\mathcal{M}_b(\Omega)$. In the linear case, however, the lack of uniqueness is avoided by using the concept of duality solution (see Section 1.5), but it is enough for the operator to be non-linear (say, $-\mathrm{div}(a(x,u,\nabla u)$, with a is a bounded function) in order to "lose" the duality definition. This problem is much more evident for operators which are nonlinear also with respect to the gradient. In this case, a further condition on the solutions has been looked for, in order to guarantee uniqueness (at least for the solutions obtained by approximation).

Definition 1.26. Let μ be a measure in $L^1(Q) + W^{-1,p'}(\Omega)$. Then $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a *entropy solution* of the problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = \mu & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.7.1}$$

if for every $k > 0$, it satisfies

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla T_k(u - \varphi)dx \le \int_\Omega T_k(u - \varphi)d\mu, \tag{1.7.2}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 1.27. Let us observe that both terms of (1.7.2) are well defined; in fact, the first one, taking into account the definition of T_k, can be rewritten as

$$\int_{\{|u|\le M\}} a(x, T_M(u), \nabla T_M(u)) \cdot \nabla T_M(T_M(u) - \varphi)dx,$$

where $M = k + \|\varphi\|_{L^\infty(\Omega)}$, now using the hypothesis (a_2), we have that $a(x, T_M(u), \nabla T_M(u)) \in (L^{p'}(\Omega))^N$, while $\nabla T_M(T_M(u) - \varphi) \in (L^p(\Omega))^N$, since $(T_M(u) - \varphi) \in \mathcal{T}_0^{1,p}(\Omega)$, for the second member of (1.7.2), we have

$$\int_\Omega T_k(u - \varphi)d\mu \le \int_{\{|u|\le M\}} T_k(T_M(u) - \varphi)d\mu + k|\mu|(\Omega)$$

and therefore it makes sense because $(T_M(u) - \varphi) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

It should be noted that there are difficulties in extending the definition of entropy solutions to the general case $\mu \in \mathcal{M}_b(\Omega)$ because of the possible lack of $\mu-$measurability of the integral on the the right-hand side of (1.7.1), however, there are cases in which this definition still makes sense outside of $L^1(\Omega) + W^{-1,p'}(\Omega)$, for example if $\mu = \delta_0$, the Dirac mass concentrated at the origin.

Remark 1.28. One can prove (see [**BGO1**]) that $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a entropy solution of problem (1.7.1) with $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla T_k(u - \varphi)dx \le \int_\Omega T_k(u - \varphi)d\mu,$$

for all $\varphi \in C_c^\infty(\Omega)$. In Definition 1.26, we can choose test functions in $C_0^\infty(\Omega)$ to obtain a equivalent problem. Finally, note that a entropy solution of problem (1.7.1), with data in $L^1(\Omega) + W^{-1,p'}(\Omega)$ is also a solution in the sense of distributions of the same problem (for the proof, see [**B6, BGO1**]).

In the rest of this part, we recall the theorem of existence and uniqueness of entropy solutions for problem (1.7.1) with measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$, in addition, we will analyze the case where $\mu = \delta_0$, in which case, Definition 1.26 still makes sense, but uniqueness is not guaranteed. To prove the uniqueness of this solution we will use the following lemma on the behavior of the *energy* of the solution u on the set where it is large, this kind of results will have a central role in our work.

LEMMA 1.29. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be an entropy solution of problem (1.7.1), with μ a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ and let us define $B_{h,k} = \{x \in \Omega : h \leq |u| \leq h+k\}$ for every $h, k > 0$. Then*

$$\lim_{h \to +\infty} \int_{B_{h,k}} |\nabla u|^p \, dx = 0.$$

PROOF. We can write $\mu = f - \operatorname{div}(F)$ with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$, then for every $h > 0$, $T_k(u) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla T_k(u - T_h(u))dx \leq \int_\Omega f T_k(u - T_h(u))dx + \int_\Omega F \cdot \nabla T_k(u - T_h(u))dx, \quad \text{for every } k > 0.$$

Now, $\nabla T_k(u - T_h(u)) = \nabla u$ in $B_{h,k}$ and it's zero elsewhere, in addition $|T_k(u - T_h(u))| \leq k$. Then we can write

$$\int_{B_{h,k}} a(x,u,\nabla u) \cdot \nabla u \; dx \leq k \int_{A_h} f dx + \int_{B_{h,k}} F \cdot \nabla u dx \leq k \int_{A_h} |f| dx + \int_{B_{h,k}} F \cdot \nabla u dx,$$

where $A_h = \{x \in \Omega : |u| \geq h\}$, using assumption ($a_1$) and the Young's inequality, we obtain

$$\alpha \int_\Omega |\nabla u|^p dx \leq k \int_{A_h} |f| dx + c \int_{B_{h,k}} |F|^{p'} dx + \frac{\alpha}{2} \int_{B_{h,k}} |\nabla u|^p dx,$$

where c is a constant depending on α, p and p'. So

$$\int_{B_{h,k}} |\nabla u|^p dx \leq \frac{2k}{\alpha} \int_{A_h} |f| dx + \frac{2c}{\alpha} \int_{B_{h,k}} |F|^{p'} dx.$$

Then, from the fact that, for $k > 0$ fixed $\operatorname{meas}(A_h) \underset{h \to +\infty}{\longrightarrow} 0$ and $\operatorname{meas}(B_{h,k}) \underset{h \to +\infty}{\longrightarrow} 0$, $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$, the result is obtained. □

Let state the main result about entropy solutions.

THEOREM 1.30. *Let μ be a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$. Then there exists a unique entropy solution of problem (1.7.1).*

PROOF. See [**BGO1**], Theorem 3.2 and Theorem 3.3. □

What happens if the datum μ is the Dirac mass concentrated at one point in Ω ? In this case the definition of entropy solution is no longer enough to guarantee its uniqueness and can be lost in the general case $\mu \in \mathcal{M}_b(\Omega)$; however, there are cases in which this definition still makes sense, even if $\mu \notin L^1(\Omega) + W^{-1,p'}(\Omega)$. First of all, We prove that if $\mu = \delta_0$ is the mass of Dirac concentrated at the origin and Ω contains the origin, then $\delta_0 \notin L^1(\Omega) + W^{-1,p'}(\Omega)$ for every $p \in [1, N)$, in this case (1.7.2) makes sense, because every function is measurable with respect to δ_0. In general the following result is true.

THEOREM 1.31. *Let $\mu \in \mathcal{M}_b(\Omega)$. Then for $p \in [1, +\infty)$ we have that $\mu \in W^{-1,p'}(\Omega)$ if and only if*

$$\int_\Omega \left[\int_0^1 \left(\frac{\mu(B(y,r))}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} \right] d\mu(y) < +\infty.$$

PROOF. See [**Zie**], Theorem 4.7.5. □

PROPOSITION 1.32. *Let $\mu = \delta_0$, $p \in [1, N)$ and Ω contains the origin of $\mathbb{R}^N$. Then $\delta_0 \notin L^1(\Omega) + W^{-1,p'}(\Omega)$.*

PROOF. Since Ω contains the origin of $\mathbb{R}^N$, we have

$$\int_\Omega \left[\int_0^1 (\frac{\delta_0(B(y,r))}{r^{N-p}})^{\frac{1}{p-1}} \frac{dr}{r}\right] d\delta_0(y) = \int_0^1 r^{\frac{1-N}{p-1}} dx;$$

applying Theorem 1.31, we have that $\delta_0 \in W^{-1,p'}(\Omega)$ (and therefore to $L^1(\Omega) + W^{-1,p'}(\Omega)$) if and only if this integral is finite, and this is true if and only if $\frac{1-N+p-1}{p-1} > 0$, or if $p > N$. The we have the desired result. ☐

As the following example shows, the case where $\mu = \delta_0$, the uniqueness fails for entropy solutions.

EXAMPLE. We are going to prove that if μ charges the sets of $p-$capacity zero, the the notion of entropy solution is not suitable in order to obtain uniqueness of solutions. Actually, let $N \geq 2$, $\Omega = B_1(0)$, and $\mu = \delta_0$ the Dirac mass concentrated in the origin of $\mathbb{R}^N$. Let us consider the following problem

$$\begin{cases} \Delta u = \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7.3}$$

It is known that (1.7.3) has a unique solution u in the sense of distributions belonging to $W_0^{1,1}(\Omega)$, it can be explicitly calculated, and is (we restrict our example to the case $N \geq 3$ for simplicity) $u(x) = C_N(|x|^{2-N} - 1)$, with C_N is a positive constant depending only on the dimension N. We are going to prove that αu is an entropy solution of (1.7.3) (that is, it satisfies (1.7.3) below) for every real number α such that $0 < \alpha \leq 1$. We begin by proving this fact for $\alpha = 1$, that is

$$\int_\Omega \nabla u \cdot \nabla T_k(u-\varphi)dx \leq \int_\Omega T_k(u-\varphi)d\delta_0 \quad \forall \varphi \in C_0^\infty(\Omega),\ \forall k > 0. \tag{1.7.4}$$

Let $f_n = \chi_{B_{\frac{1}{n}}}/\text{meas}(B_{\frac{1}{n}}(0))$, as it is well know, f_n converges to δ_0 in the weak–* topology of measures. Let u_n be the solution of

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By the results of [**BG1**], $\{u_n\}$ converges to u in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. On the other hand, it is easy to see (also u_n can be explicitly calculated) that u_n is greater that $C_n(|n|^{2-N} - 1)$ on $B_{\frac{1}{n}}(0)$, so that, for *fixed* k and φ

$$\int_\Omega f_n T_k(u_n - \varphi)dx = k = \int_\Omega T_k(u-\varphi)d\delta_0,$$

observe that the last expression has sense because $T_k(u-\varphi)$ is continuous. Moreover, using the explicit expression of u_n, for every fixed $k > 0$, there exists $n(k) \in \mathbb{N}$ such that $T_k(u_n)$ is equal to $T_k(u)$ for every $n > n(k)$. Thus, using properties of u_n, and recalling that the test functions φ are bounded,

$$\lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla T_k(u_n - \varphi)dx = \int_\Omega \nabla u \cdot \nabla T_k(u-\varphi)dx,$$

and so u is an entropy solution of (1.7.3), in the sense that (1.7.3) holds with "$\leq$" replaced by "$=$". Note that this fact is true only for δ_0, but also for any other datum of the form δ_a, $a \in \Omega$ (and for the corresponding usual weak solution of (1.7.3)). Let now α be a real number in $(0,1)$. Then, since u is an entropy solution,

$$\int_\Omega \nabla(\alpha u) \cdot \nabla T_k(\alpha u - \varphi)dx = \alpha^2 \int_{\{|u-\frac{\varphi}{\alpha}| \leq \frac{k}{\alpha}\}} \nabla u \cdot \nabla(u - \frac{\varphi}{\alpha})dx \leq \alpha^2 \frac{k}{\alpha} \leq k,$$

and so αu is an entropy solution of (1.7.3), i.e., the entropy solution is not unique. Observe that αu is not a solution in the distribution sense of (1.7.3) if $\alpha \neq 1$.

One can think that there exists at most a unique function u that satisfies (1.7.3) with "$\leq$" replaced by "$=$". This is not true, actually, let u as before, and let v be the solution of equation (1.7.3) where δ_0 is replaced by δ_a, with $a \in \Omega$, $a \neq 0$. Then $u_\theta = \theta u + (1-\theta)v$ is a solution of (1.7.3) with "$\leq$" replaced by "$=$" for every $\theta \in (0,1)$. This gives infinitely many solutions.

However, if μ belongs to $\mathcal{M}_b(\Omega)$, then the entropy solution is also a solution in the sense of distributions, as we said before. In other works, the behaviour of a solution u around its blow-up points (behaviour that is not considered in the formulation of entropy solutions), turns out to be unimportant if μ does not charge the sets of zero $p-$capacity, but it has to be considered if this is not the case.

1.8. Renormalized solutions

Let us come back to the problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8.1}$$

whether assumptions $(a1) - (a3)$ are necessary in order to have existence of solutions via a stability method described in Theorem 1.22, it would be desirable to have a notion of solution which is much involved with the stability properties of the equations and such that we still have existence. In order to answer this question we introduce here the definition of renormalized solution of (1.8.1) extending the notion developed in Section 1.7.

In fact, the definition of renormalized solution was first given in [**DL1, DL2**] in the context of hyperbolic equations of conservation laws and then adapted to second order elliptic problems in [**BDGM**]. In the theory of boundary value problems with L^1-data this notion has been recently used in order to get uniqueness of solutions at least for data in $L^1(\Omega)$ [**LM**]. Following a recent extension of this framework to general measure data $\mu \in \mathcal{M}_b(\Omega)$ provided in [**DMOP**], we recall how, in dealing with problem (1.8.1), the renormalized solutions emphasize the stability properties mentioned above by selecting suitable test functions. Roughly speaking, the idea of renormalized solutions is to multiply the equation solved by u by test functions using $S(u)$, where S is in $W^{1,\infty}(\mathbb{R})$ and has compact support, so that the equation is in some sense reduced to the subset of Ω where $|u| \le M$, where M is such that $\mathrm{supp}(S) \subset [-M, M]$, and u can be replaced by its truncation $T_M(u)$, which belongs to the energy space $W_0^{1,p}(\Omega)$. The meaning of the term $S(u)\mu$ is then motivated by the fact that

$$\lim_{n\to\infty} \int_\Omega \lambda_n S(u_n)\varphi dx = S(+\infty) \int_\Omega \varphi d\lambda \quad \forall \varphi \in C_c^\infty(\Omega),$$

where $S(+\infty) = \lim_{t\to+\infty} S(t)$ (this limit exist and is finite since S' has compact support) and $S(u)\lambda = 0$ (since S is also with compact support) and from the fact that, being $S(u) = S(T_M(u))$ the measure μ_0 may be applied to $S(u)$, both in the sense of measures (see *(2.J)*) and in the sense of $L^1(\Omega) + W^{-1,p'}(\Omega)$. Then (1.8.1) is transformed into the renormalized equation:

$$-\mathrm{div}(S(u)a(x,u,\nabla u)) + S'(u)a(x,u,\nabla u)\cdot\nabla u = S(u)\mu_0. \tag{1.8.2}$$

On the other hand, since the equation (1.8.2) only considers the properties of the truncations of u, the renormalized formulation usually needs to add an extra condition to recover, in some sense, the behaviour of u at infinity. Moreover (1.8.2) does not take into account the singular part λ in the decomposition of the measure μ, so that λ has to be related to this extra condition at infinity. Let us then introduce the definition of renormalized solution we will use hereafter. We give this definition for signed (singular) measures in the spirit of [**DMOP**].

DEFINITION 1.33. Let $\mu \in \mathcal{M}_b(\Omega)$ be splitted as in Theorem 1.20, that is:

$$\mu = \mu_0 + \lambda = f - \mathrm{div}(F) + \lambda^+ - \lambda^-$$

A function u in $\mathcal{T}_0^{1,p}(\Omega)$ is said to be a renormalized solution of (1.8.2) if for every S in $W^{1,\infty}(\mathbb{R})$ having compact support we have:

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(S(u)\varphi)dx = fS(u)\varphi dx + \int_\Omega F\cdot\nabla(S(u)\varphi)dx \quad \forall\varphi\in W_0^{1,p}(\Omega), \tag{1.8.3}$$

and

$$\begin{aligned} &\lim_{n\to+\infty}\int_{\{x:n\le u\le n+1\}} a(x,u,\nabla u)\cdot\nabla u\varphi\, dx = \int_\Omega \varphi d\lambda^+ \quad \forall\varphi\in C(\overline{\Omega}), \\ &\lim_{n\to+\infty}\int_{\{x:-n-1\le u\le -n\}} a(x,u,\nabla u)\cdot\nabla u\varphi\, dx = \int_\Omega \varphi d\lambda^- \quad \forall\varphi\in C(\overline{\Omega}), \end{aligned} \tag{1.8.4}$$

where $\lambda^\pm$ denote the positive and negative part of the measure λ.

Note that all the integrals appearing in the renormalized formulation are well defined since S has compact support and $T_k(u)$ belongs to $W^{1,p}(\Omega)$ for every $k > 0$. In order to understand how (1.8.4) appears, it is important to recall some others definitions of renormalized solutions if the data is general. We will say that a function $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies condition (1.8.2) if there exists $k > 0$ and two functions $w^{+\infty}, w^{-\infty} \in C_b^1(\Omega)$, such that

$$\begin{cases} w = w^{+\infty} & \text{a.e. in } \{u > k\}, \\ w = w^{-\infty} & \text{a.e. in } \{u < -k\}. \end{cases} \tag{1.8.5}$$

Definition 1.34. Let $\mu \in C_b^1(\Omega)$. A function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a renormalized solution of problem (1.8.1), if the following conditions hold

(a) $|\nabla u|^{p-1} \in L^q(\Omega) \qquad \forall q < \frac{N}{N-1}$,

(b) for any $w \in W_0^{1,p}(\Omega)$ that satisfies condition (1.8.2), then

$$\int_\Omega a(x,u,\nabla u).\nabla w \; dx = \int_\Omega w d\mu_0 + \int_\Omega w^{+\infty} d\mu_s^+ - \int_\Omega w^{-\infty} d\mu_s^- . \tag{1.8.6}$$

Remark 1.35. Notice that all terms in (1.8.6) are well defined, in fact, as far as the first term is concerned, it can be written as

$$\int_{\{|u|\le k\}} a(x,u,\nabla u) \cdot \nabla w \; dx + \int_{\{|u|>k\}} a(x,u,\nabla u) \cdot \nabla w \; dx, \tag{1.8.7}$$

for $k > 0$ and $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying condition (1.8.2), so

$$\int_{\{|u|\le k\}} a(x,u,\nabla u) \cdot \nabla w \; dx = \int_{\{|u|\le k\}} a(x,u,\nabla T_k(u)) \cdot \nabla w \; dx, \tag{1.8.8}$$

is well defined since, thanks to assumption (a$_2$), $a(x, T_k(u), \nabla T_k(u)) \in (L^{p'}(\Omega))^N$ and $\nabla w \in (L^p(\Omega))^N$ on the other hand, the second term of (1.8.8) makes sense since, w satisfy assumption (1.8.2), and so $\nabla w \in L^\infty(\{|u| > k\})$ while $a(x,u,\nabla u) \in (L^q(\Omega))^N$ for any $q < \frac{N}{N-1}$. The right hand side of (1.8.6) makes sense as well, since, using Theorem 1.20 $\int_\Omega w d\mu_0$ is well defined because of the fact that $w \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, while there are no problem to give sense at the last two terms of (1.8.6), since $w^{+\infty}$ and $w^{-\infty}$ are two bounded and continuous functions on Ω. Let us also observe that we can choose in (1.8.6) the functions $w \in C_0^\infty(\Omega)$ (with $w^{+\infty} = w^{-\infty} = w$), and so a renormalized solution turns out to be a distributional solution of problem (1.8.1).

As we mentioned above, a renormalized solution turns out to coincide with an entropy solution if $\mu \in \mathcal{M}_0(\Omega)$; actually we can easily prove the following result

Proposition 1.36. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then, problem* (1.8.1) *has at most one renormalized solution.*

Proof. Thanks to 1.30, it will be enough to prove that, if u is a renormalized solution of problem (1.8.1), then u is an entropy solution of the same problem. For any $h > 0$, we can choose in (1.8.6), $w = T_h(u - \varphi)$, with $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$; in fact, we have

$$w = T_h(T_{h+M}(u) - \varphi),$$

where $M = \|\varphi\|_{L^\infty(\Omega)}$, and so $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, w satisfy condition (1.8.6) since we can choose $w^{+\infty} = h$, $w^{-\infty} = -h$ and $k = h + M$. Hence, using $w = T_h(u - \varphi)$ in (1.8.6) one can readily check that u is an entropy solution (with equality sing) being $\mu_s^+ = \mu_s^- = 0$. ☐

In order to obtain this existence result of at least one renormalized solution of problem (1.8.1) when μ is an arbitrary measure of $\mathcal{M}_b(\Omega)$, the key point is to prove the strong convergence in $W_0^{1,p}(\Omega)$ of the truncations at every fixed height k of the solutions of problem (1.8.1) corresponding to some (special but fairly general) approximations of μ. (This is actually the result of continuity with respect to μ to which we made allusions above.) This continuity result is proved by means of a careful study of the energies of the truncations of these solutions "far" and "near" the set where the measure λ is concentrated. Concerning uniqueness results, which in particular allow to recover the uniqueness of the renormalized (or entropy) solution of problem (1.8.1) in the particular case where μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. In the case of an arbitrary measure of $\mathcal{M}_b(\Omega)$, one of uniqueness results is the following one: let u and $\tilde{u}$ be two renormalized solutions of problem (1.8.1), if $u - \tilde{u}$ belongs to $L^\infty(\Omega)$ (this condition can be replaced by weaker ones), then $u = \tilde{u}$.

1.9. Elliptic equations with absorption term

The linear theory is an important tool to understand the nonlinear Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Some pioneering contributions to nonlinear problems with L^1 or measure data are due to Brezis and Strauss [**BS**], Lieb and Simon [**LS**] and Bénilan with Brezis (see [**BB, Br2, Br3**]). According to Stampacchia's regularity theory, every solution of the linear Dirichlet problem belongs to the Sobolev spaces $W_0^{1,q}(\Omega)$ for $1 \le q < \frac{N}{N-1}$ (see Section 1.5), This is an important difference with respect to the Calderón-Zygmund L^p-theory which tells that if $\mu \in L^p(\Omega)$ for some $1 < p < +\infty$, then the solution of the linear Dirichlet problem belongs to $W^{2,p}(\Omega)$. The motivation for studying such problems is beautifully discussed in the preface of [**BB**] by H. Brezis. The study of the nonlinear Dirichlet problem with measure data turns out to be more subtle than with L^1-data. It was observed in [**BB, Br2, Br3**] that if $N > 3$ and $g(t) = |t|^{p-1}t$, with $p \ge \frac{N}{N-2}$, then the nonlinear Dirichlet problem has no solution when μ is a Dirac mass. They also proved that if $p < \frac{N}{N-2}$ and $N \ge 2$, the nonlinear Dirichlet problem has a solution for any finite measure μ. Later, Baras and Pierre [**BPi**] characterized all measures μ for which the nonlinear Dirichlet problem admits a solution for a nonlinearity of the form $g(t) = |t|^{p-1}t$. Their necessary and sufficient condition for the existence of a solution when $p \ge \frac{N}{N-2}$ can be expressed in terms of the $W^{2,p'}-$capacity. The case of exponential nonlinearities of the form $g(t) = e^t - 1$ was studied by Vázquez [**Va1**] in dimension $N = 2$ and more recently by [**BLOP**] in dimension $N \ge 3$. The solution in this case is related to the Hausdorff measure $\mathcal{H}^{N-2}$.

Brezis, Marcus and Ponce [**BrMP1**] introduced the concept of reduced measure in order to analyze the nonexistence mechanism behind the nonlinear Dirichlet problem and to describe what happens if one forces the problem to have a solution in cases where the problem refuses to have one. The approach developed in [**BrMP1**] was to introduce an approximation scheme. For example, the measure μ is kept fixed and g is truncated. Alternatively, the nonlinearity g is kept fixed and μ is approximated via convolution. It was originally observed by Brezis [**Br1**] that if $N \ge 3$, $g(t) = |t|^{p-1}t$, with $p \ge \frac{N}{N-2}$, and μ is a Dirac mass, then all natural approximations u_n of the nonlinear Dirichlet problem converge to 0. However, 0 is not a solution corresponding to a Dirac mass.

Now let us consider the problem

$$\begin{cases} -\Delta u + |u|^{q-1}u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 3$, is a bounded smooth domain, $1 < q < \infty$, and μ is a bounded Radon measure on Ω. A function $u \in L^q(\Omega)$ is called *weak* solution of (1.9.2) if

$$-\int_\Omega u\Delta\varphi \, dx + \int_\Omega |u|^{q-1}u\varphi \, dx = \int_\Omega \varphi d\mu \quad \forall \varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega.$$

It is known [**S, BrMP1**] that a weak solution u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{(N-1)}$. The celebrated result by Bénilan and Brézis [**Br1**] states that if μ is the Dirac mass at a point of Ω, then in the case $q < \frac{N}{N-2}$ there exists a unique weak solution. Moreover, if $q \ge \frac{N}{N-2}$, distributional solutions in $L^q_{loc}(\Omega)$ do not exist. It is to be noted here that when $\mu \in L^1(\Omega)$ the problem (1.9.2) admits a unique solution in some appropriate class without any restriction on q.

The phenomenon of the non-existence can be better understood using the notion of capacity [**BrMP1, BrMP2**]. Roughly speaking, given an exponent q, if the measure μ on the right-hand side is concentrated on a very "small" set, then distributional solutions do not exist. [**BPi**] (see also Galouët and Morel [**GM**]) were able to characterize how much such set must be small, in terms of q, in order to obtain non-existence of distributional solutions. Namely, they proved that a distributional solution $u \in L^q(\Omega) \cap W_0^{1,1}(\Omega)$ exists if and only if

$$|\mu|(E) = 0 \text{ for every Borel set } E \subset \Omega \text{ with } \mathrm{cap}_{2,q'}(E) = 0,$$

where $\mathrm{cap}_{2,q'}$ denotes that capacity associated to $W_0^{2,q'}$ (see Chapter 7). This result is consistent with that only if $q \geq \frac{N}{N-2}$ (see, Meyers [**Mey**]).

The failure of existence discussed above can be seen also from another point of view. Suppose that $q \geq \frac{N}{N-2}$, Let us consider first the case $\mu = f \in L^1(\Omega)$ (case of existence), let $f_n \in L^1(\Omega)$ be a sequence of functions converging to f in the sense of measures, and consider the problem (1.9.2) with μ replaced by f_n. Such problem admits a unique solution u_n [**BS**], and the sequence u_n converges to u, where u is the solution when the datum is f. Consider now the case $\mu = \delta$, where δ is the Dirac mass at a point of Ω, say 0 (case of non-existence). Setting for instance $f_n = \chi_{B(0,\frac{1}{n})}|_{|B(0,\frac{1}{n})|}$, we have $f_n \to \delta$, and proceeding analogously, one gets $u_n \to 0$. Notice that the function identically zero is not a solution of (1.9.2) (see [**Br1, BV**] for details). The fact that in this case solutions do not exist can be roughly expressed saying that sequences of solutions of approximating equations do not converge to a reasonable solution.

1.10. Functional parabolic spaces

Given a real Banach space V, we will denote by $C^\infty(\mathbb{R}; V)$ the space of functions $u : R \to V$ which are infinitely many times differentiable (according to the definition of Fréchet differentiability in Banach spaces) and by $C_c^\infty(\mathbb{R}; V)$ the space of functions in $C^\infty(\mathbb{R}; V)$ having compact support. For a, b in $\mathbb{R}$, $C_c^\infty([a, b]; V)$ will be the space of restrictions to $[a, b]$ of functions of $C_c^\infty(\mathbb{R}; V)$, and $C([a, b]; V)$ the space of continuous functions from $[a, b]$ into V. Then for $1 \leq p < +\infty$, $L^p(a, b; V)$ is the space of measurable functions $u : [a, b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p dt\right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(a, b; V)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(a,b;V)} = \sup_{[a,b]} \text{ess } \|u\|_V < +\infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [**DL1**], Chapter XVIII.

Let us recall that, for $1 \leq p \leq \infty$, $L^p(a, b; V)$ is a Banach space, moreover if $1 \leq p < \infty$ and V', the dual space of V, is separable, then the dual space of $L^p(a, b; V)$ can be identified with $L^{p'}(a, b; V')$. Now, given a function u in $L^p(a, b; V)$, it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a, b; V)$, which is the space of linear continuous functions from $C_c^\infty(a, b)$ into V [**Sc**]. In fact, the definition is the following

$$\langle u_t, \psi\rangle = -\int_a^b u\psi_t \, dt \qquad \forall \psi \in C_c^\infty(a, b),$$

where the equality is meant in V. If u belongs to $C^1(a, b; V)$ this definition clearly coincides with the Fréchet-derivative of u. In the following, when u_t is said to belong to a space $L^q(a, b; \tilde{V})$ ($\tilde{V}$ being a Banach space) this means that there exists a function z in $L^q(a, b; \tilde{V}) \cap \mathcal{D}'(a, b; V)$ such that:

$$\langle u_t, \psi\rangle = -\int_a^b u\psi_t \, dt = \langle z, \psi\rangle \qquad \forall \psi \in C_c^\infty(a, b).$$

In the following, we will also use the notation $\frac{\partial u}{\partial t}$ instead of u_t sometimes. We recall the following classical embedding result (see [**DL1**], Chapter XVIII, Section 2, Theorem 1). Let H be an Hilbert space such that

$$V \underset{dense}{\subseteq} H \subseteq V', \tag{1.10.1}$$

let u in $L^p(a, b; V)$ be such that u_t, defined as above in distributional sense, belongs to $L^{p'}(a, b; V')$. Then u belongs to $C([a, b]; H)$. This result also allows to deduce, for functions u and v enjoying these properties, the integration by parts formula

$$\int_a^b \langle v, u_t\rangle dt + \int_a^b \langle u, v_t\rangle dt = (u(b), v(b)) - (u(a), v(a)), \tag{1.10.2}$$

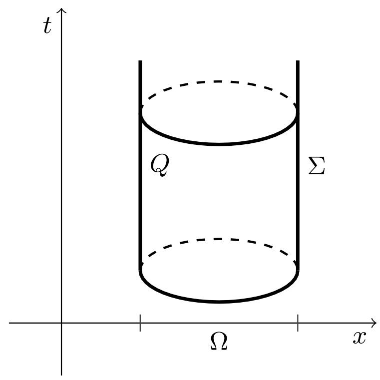


FIGURE 3. Parabolic boundary domain

where here $\langle \cdot, \cdot \rangle$ is the duality between V and V' and $(\cdot, \cdot)$ the scalar product in H. Note that (1.10.2) makes sense thanks to the embedding result previously mentioned. Its proof relies on the fact that $C_c^\infty([a, b]; V)$ is dense in the space of functions $u \in L^p(a, b; V)$ such that u_t belongs to $L^{p'}(a, b; V')$ endowed with the norm $\|u\| = \|u\|_{L^p(a,b;V)} + \|u_t\|_{L^{p'}(a,b;V')}$, together with the fact that (1.10.2) is true for u, v in $C_c^\infty([a, b]; V)$ by the theory of integration and derivation in Banach spaces. Note however that in this context (1.10.2) is subject to the verification of (1.10.1), if for instance $V = W_0^{1,p}(\Omega)$, then (1.10.1) is true with $H = L^2(\Omega)$ but only if $p \geq \frac{2N}{N+2}$. We will see in the next section a possible extensions for more parabolic initial boundary value problems in generalized context of divergence form operators.

1.11. Parabolic operators on classical Sobolev spaces

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary, p and p' be a real numbers, with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, $|\zeta|$ and $\zeta \cdot \zeta'$ will denote respectively the Euclidean norm of a vector $\zeta \in \mathbb{R}^N$ and the scalar product between ζ and $\zeta' \in \mathbb{R}^N$. Let $T > 0$, and let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$. We will denote by Q the cylinder $\Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ its lateral surface. Let then $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e. measurable with respect to (t, x) for every fixed (s, ζ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ζ) for almost every fixed (t, x) in Q) such that there exists $p > 1$ for which the following assumptions hold true

$$a(t, x, s, \zeta) \cdot \zeta \geq \alpha |\zeta|^p \quad \alpha > 0, \tag{1.11.1}$$

$$|a(t, x, s, \zeta)| \leq \beta(k(t, x) + |s|^{p-1} + |\zeta|^{p-1}) \quad \beta > 0, \quad k(t, x) \in L^{p'}(Q), \tag{1.11.2}$$

$$(a(t, x, s, \zeta) - a(t, x, s, \eta)) \cdot (\zeta - \eta) > 0, \tag{1.11.3}$$

for every ζ, η ($\zeta \neq \eta$) in $\mathbb{R}$ and almost every (t, x) in Q. Thanks to (1.11.1) – (1.11.3), it is possible to define on the space $L^p(0, T; W_0^{1,p}(\Omega))$ the operator $A(u) = -\operatorname{div}(a(t, x, u, \nabla u))$, which then maps $L^p(0, T; W_0^{1,p}(\Omega))$ into $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and is bounded and coercive.

Given f in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and u_0 in $L^2(\Omega)$, by a weak solution of

$$\begin{cases} u_t - \operatorname{div}(a(t, x, u, \nabla u)) = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.11.4}$$

we mean a function u in $L^p(0, T; W^{1,p}(\Omega))$ which satisfies the equation (1.11.4) in the sense of distributions, that is

$$-\int_Q u\psi_t\varphi \, dxdt + \int_Q a(t, x, u, \nabla u) \cdot \nabla\varphi\psi \, dxdt = \int_0^T \langle f, \varphi \rangle \psi \, dt \quad \forall \psi \in C_c^\infty(0, T) \quad \forall \varphi \in C_c^\infty(\Omega), \tag{1.11.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. As a consequence of the equation and (1.11.2) we deduce that u_t (which initially only belongs to $\mathcal{D}'(a, b; W_0^{1,p}(\Omega))$) in fact belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and it follows that

$$\int_0^T \langle u_t, v\rangle dt + \int_Q a(t, x, u, \nabla u) \cdot \nabla v \; dxdt = \int_0^T \langle f, v\rangle dt \quad \forall v \in L^p(0, T : W_0^{1,p}(\Omega)).$$

Moreover from the injection result previously mentioned, if $p \geq \frac{2N}{N+2}$ then u belongs to $C([0, T]; L^2(\Omega))$, which gives a meaning to the initial condition $u(0)$ (i.e. $u(0) = u_0$ in $L^2(\Omega)$). Nevertheless, even if $p < \frac{2N}{N+2}$, it is possible to find a weak solution u of (1.11.4) which belongs to $C([0, T]; L^2(\Omega))$, as stated in the following classical result by J. Leray and J.-Louis Lions.

THEOREM 1.37. *Let* (1.11.1) $-$ (1.11.3) *hold true, and let f be in $L^{p'}(0, T; W^{-1,p'}(\Omega))$. Then there exists a weak solution u in $L^p(0, T; W_0^{1,p}(\Omega)) \cap C([0, T]; L^2(\Omega))$ of* (1.11.4).

PROOF. See [**L**]. □

REMARK 1.38. The equation appearing in (1.11.4) can be considered both in the space of vector valued distributions, as we did before in (1.11.5), and in the space of distributions in Q, that is

$$-\int_Q u\frac{\partial \zeta}{\partial t} \; dxdt + \int_Q a(t, x, u, \nabla u) \cdot \nabla \zeta \; dxdt = \int_0^T \langle f, \zeta\rangle \quad \forall \zeta \in C_c^\infty(\Omega \times (0, T)). \tag{1.11.6}$$

1.12. Parabolic capacity and Measures

Let us recall that the parabolic initial boundary value problem, that is (1.11.4) with $u_0 = 0$, was studied first in [**BG1**] under the general assumptions that μ and u_0 are bounded Radon measures respectively on Q and on Ω. In this case it was proved the existence of a weak solution u which only belongs to the space $L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ and it was also asked that $p > 2 - \frac{1}{N+1}$ in order to have that $p - \frac{N}{N+1} > 1$ and the equation can be considered in a weak sense (the weak formulation also contains the initial condition). The evolution equation with integrable data was then considered in many other later papers [**Pr2, BM, BDGO, AMST**], especially for questions concerning uniqueness of solutions. Thanks to assumptions (1.11.1) $-$ (1.11.3), in [**DO2**] it was proved that, even with $\mu \in L^1(Q)$ and lower order term, there exists a weak solution of (1.11.4) belonging to $L^p(0, T; W_0^{1,p}(\Omega))$ (and the extension to the case if μ belongs to $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$ was given in [**DPP**]). This is consistent with the content of Theorem 1.11 for elliptic equations which pointed out that (1.11.1) $-$ (1.11.3) (or, better, the equivalent assumptions in the elliptic case) allows to find solutions having finite energy. Let us recall that a fundamental notions on capacity made in the context of elliptic equations with measures as right hand side (written in Section 1.4) exists for data in $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$, as consequence of the equation and the a priori estimates. This was also the first motivation for our research in Chapters 3, 5 and in Chapters 4, 6 for the general case. We start by a description of the functional spaces needed for capacities.

DEFINITION 1.39. Let $T > 0$ and $p > 1$. The capacity Sobolev space

$$W = \{u \in L^p(0, T; V),\; u_t \in L^{p'}(0, T; V')\} \tag{1.12.1}$$

is a Banach space endowed with the norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$, where $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$.

Let us define also, for every $p > 1$, the space S^p as

$$S^p = \{u \in L^p(0, T; W_0^{1,p}(\Omega)),\; u_t \in L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))\} \tag{1.12.2}$$

endowed with its natural norm $\|u\|_{S^p} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$, it is clear that $S^p \underset{\text{inj cont}}{\hookrightarrow} C(0, T; L^1(\Omega))$ and its subspace W_2 as

$$W_2 = \{u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q),\; u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)\}$$

endowed with its natural norm $\|u\|_{W_2} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u\|_{L^\infty(Q)} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$.

Let us define for every Borel set $B \subseteq Q$, its $p-$capacity $\mathrm{cap}_p(B, Q)$ with respect to Q by

$$\inf \{\|u\|_W\}$$

where the infimum is taken over all the functions $u \in W$ such that $u \geq 1$ almost everywhere in a neighborhood of B. We recall also the other notion of parabolic $p-$capacity associated to our problem (for further details, see **[P, DPP]**).

Definition 1.40. Let T be a real number, with $T > 0$, let K be a compact subset of Q. The parabolic $p-$capacity of K with respect to Q is defined as

$$\mathrm{cap}_p(K, Q) = \inf \{\|u\|_W : \ u \in C_c^\infty(Q), \ u \geq \chi_K\}, \tag{1.12.3}$$

where χ_K is the characteristic function of K, we will use the convention that $\inf \emptyset = +\infty$. The parabolic $p-$capacity of any open subset of Q is then defined by

$$\mathrm{cap}_p(U, Q) = \sup \{\mathrm{cap}_p(K, Q), \ K \text{ compact}, \ K \subset Q\}, \tag{1.12.4}$$

and the parabolic $p-$capacity of any Borelian set $B \subset Q$ by

$$\mathrm{cap}_p(B, Q) = \inf \{\mathrm{cap}_p(U, Q), \ U \text{ open}, \ B \subset U\}. \tag{1.12.5}$$

We say that a property $\mathcal{P}(t, x)$ holds cap_p quasi-everywhere if $\mathcal{P}(t, x)$ holds for every (t, x) outside a subset of Q of zero $p-$capacity. A function u defined on Q is said to be cap_p quasi-continuous if for every $\epsilon > 0$ there exists $B \subseteq Q$ with $\mathrm{cap}_p(B, Q) < \epsilon$ such that the restriction of u to $Q \backslash B$ is continuous. It is well known that every function in W has a unique cap_p quasi-continuous representative, whose values are defined cap_p quasi-everywhere in Q **[DPP, Pe1]**. In what follows we always identify a function $u \in W$ with its cap_p quasi-continuous representative. A set $E \subseteq Q$ is said to be cap_p quasi-open if for every $\epsilon > 0$, there exists an open set U such that $E \subseteq U \subseteq Q$ and $\mathrm{cap}_p(U \backslash E, Q) \leq \epsilon$. It can be easily seen that, if u is a cap_p quasi-continuous function, then for every $k \in \mathbb{R}$ the sets $\{u > k\} = \{(t, x) \in Q : u(t, x) > k\}$ and $\{u < k\} = \{(t, x) \in Q : u(t, x) < k\}$ are cap_p quasi-open. The characteristic function of a cap_p quasi-open set can be approximated by a monotonic sequence of functions in the energy space W, as stated in the following Lemma.

Lemma 1.41. *For every cap_p quasi-open set $E \subseteq Q$ there exists an increasing sequence (w_n) of non-negative functions in W which converges to χ_E cap_p quasi-everywhere in Q.*

Proof. See **[PPP2]**, Lemma 2.1. □

We define $\mathcal{M}_b(Q)$ as the space of all Radon measures on Q with bounded total variation, and $C_b(Q)$ as the space of all bounded, continuous functions on Q, so that $\int_Q \varphi d\mu$ is defined for $\varphi \in C_b(Q)$ and μ in $\mathcal{M}_b(Q)$. The positive part, the negative part, and the total variation of a measure μ in $\mathcal{M}_b(Q)$ are denoted by μ^+, μ^-, and $|\mu|$, respectively. We recall that for a measure μ in $\mathcal{M}_b(Q)$, and a Borel set $E \subseteq Q$, the measure $\mu \perp E$ is defined by $(\mu \perp E)(Q) = \mu(E \cap B)$ for any Borel set $B \subseteq Q$. We define $\mathcal{M}_0(Q)$ as the set of all measures μ in $\mathcal{M}_b(Q)$ which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq Q$ such that $\mathrm{cap}_p(B, Q) = 0$, while $\mathcal{M}_s(Q)$ will be the set of all measures μ in $\mathcal{M}_b(Q)$ for which there exists a Borel set $B \subset Q$, with $\mathrm{cap}_p(B, Q) = 0$, such that $\mu = \mu \perp E$.

Remark 1.42. It can be easily seen that μ belongs to $\mathcal{M}_0(Q)$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(B) < \epsilon$ for every Borel set $B \subseteq Q$ with $\mathrm{cap}_p(B, Q) < \delta$.

In order to obtain more precise convergence results, we need the following characterization of the measures in $\mathcal{M}_0(Q)$.

Lemma 1.43. *A measure μ_0 belongs to $\mathcal{M}_0(Q)$ if it belongs to $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^p(0, T; V)$. Thus, if $\mu_0 \in \mathcal{M}_0(Q)$, there exist $f \in L^1(Q)$, $F \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $g_t \in L^p(0, T; V)$ such that $\mu_0 = f + F + g_t$ in the sense of distributions. Moreover*

$$\int_Q \varphi d\mu = \int_Q f\varphi \ dxdt + \int_0^T \langle F, \varphi \rangle dt - \int_0^T \langle \varphi_t, g \rangle dt. \tag{1.12.6}$$

for any $\varphi \in C_c^\infty([0, T] \times \Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between V' and V.

Proof. See **[DPP]**, Theorem 1.1. □

So, if μ is in $\mathcal{M}_b(Q)$, thanks to a well known decomposition result, see for instance **[FST]**, we can split it into a sum (uniquely determined) of its absolutely continuous part μ_0 with respect to the $p-$capacity and its singular part μ_s, that is μ_s is concentrated on a set E of zero $p-$capacity; we will say that $\mu_s \perp \mathrm{cap}_p$. Hence, if $\mu \in \mathcal{M}_s(Q)$, by Lemma 1.43, we have the next result

THEOREM 1.44. *For every $\mu \in \mathcal{M}_b(Q)$, there exists a unique pair (μ_0, μ_s) such that $\mu = \mu_0 + \mu_s$, $\mu_0 \in \mathcal{M}_0(Q)$, $\mu_s \in \mathcal{M}_s(Q)$ and*

$$\mu = f - \mathrm{div}(G) + g_t + \mu_s^+ - \mu_s^- \,, \tag{1.12.7}$$

in the sense of distributions, for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T;W_0^{1,p}(\Omega))$, where μ_s^+ and μ_s^- are respectively the positive and the negative parts of μ_s.

PROOF. See **[FST]**, Lemma 2.1. □

We say that a sequence (μ_n) of measures in $\mathcal{M}_b(Q)$ converges in the narrow topology to a measure μ in $\mathcal{M}_b(Q)$ if

$$\lim_{n\to+\infty} \int_\Omega \varphi d\mu_n = \int_\Omega \varphi d\mu, \tag{1.12.8}$$

for every $\varphi \in C_b(Q)$. If (1.12.8) holds only for all the continuous functions φ with compact support in Q (i.e., $\varphi \in C_c(Q)$), then we have the usual weak–* convergence in $\mathcal{M}_b(Q)$.

REMARK 1.45. It can be easily seen that a sequence of non-negative measures (μ_n) converges to μ in the narrow topology if and only if it converges to μ in the weak–* topology and the measures $\mu_n(\Omega)$ converges to $\mu(\Omega)$. Hence, for non-negative measures, the narrow convergence is equivalent to the convergence in (1.12.8) for every $\varphi \in C^\infty(\overline{Q})$.

An easy consequence of the dominated convergence theorem is the following result.

PROPOSITION 1.46. *Let μ_0 be a measure in $\mathcal{M}_0(Q)$, and let u be a function in W_2. Then u is measurable with respect to μ_0. If u further is in $L^\infty(Q)$, then u belongs to $L^\infty(Q,\mu_0)$, hence to $L^1(Q,\mu_0)$, and $\|u\|_{L^\infty(Q,\mu_0)} = \|u\|_{L^\infty(Q)}$.*

PROOF. The proof can be performed arguing as in **[DMOP]** and **[HKM]** for the elliptic case. □

Note that this property will be often used in what follows.

REMARK 1.47. Let (ρ_ϵ) be a sequence of functions in $L^1(Q)$ that converges to ρ weakly in $L^1(Q)$, and let (σ_ϵ) be a sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to σ almost everywhere on Q. Then, as a consequence of Egorov's theorem

$$\lim_{\epsilon\to 0} \int_Q \rho_\epsilon \sigma_\epsilon \; dxdt = \int_Q \rho\sigma \; dxdt \tag{1.12.9}$$

We consider now the initial boundary value problem

$$\begin{cases} u_t - \mathrm{div}(a(t,x,u,\nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.12.10}$$

where the data are taken such that

$$\mu \in \mathcal{M}_b(Q), \quad u_0 \in L^1(\Omega), \tag{1.12.11}$$

where $\mathcal{M}_b(Q)$ is the space of Radon measures on Q with bounded total mass (i.e., $\mu(Q) < +\infty$). As in the elliptic case of Section 1.4, the relationship between the possibility to find solution of (1.12.10) and the stability properties is a density argument, that is approximating the singular data μ and u_0 with sequences of

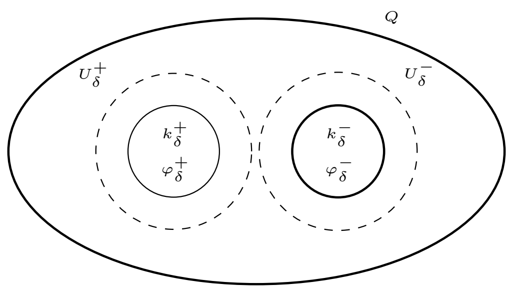


FIGURE 4. The contruction of cut-off functions

smooth functions. For example, letting (μ_n) and u_0^n be a standard approximation of μ and u_0 constructed by convolution, satisfying the following conditions

$$\begin{aligned}
&\mu_n = \mu_0^n + \mu_s^n = f_n - \operatorname{div}(G_n) + g_t^n + \lambda_n^{\oplus} - \lambda_n^{\ominus} \\
&\mu_0^n \in \mathcal{M}_0(Q), \quad \mu_s^n = \mu_s^n \perp E \text{ with } \operatorname{meas}(E) = 0 \\
&\mu_0^n \in C_c^\infty(\Omega), \quad u_0^n \to u_0 \text{ strongly in } L^1(\Omega) \\
&\mu_n \in C^\infty(Q)_c, \quad \mu_0^n + u_s^n \to \mu \text{ tigntly in } Q, \text{ i.e., in the sense of (1.12.11).}
\end{aligned} \tag{1.12.12}$$

We also have that the sequence of functions (f_n) weakly converges to f in $L^1(Q)$, G_n strongly converges in $L^{p'}(Q)^N$ and g_t^n converges in $L^p(0,T;V)$. Then there exist a sequence (u_n) of solutions of the Cauchy-Dirichlet problems

$$\begin{cases}
(u_n)_t - \operatorname{div}(a(t,x,u_n,\nabla u_n)) = \mu_n & \text{in } Q, \\
u_n = 0 & \text{in } \Sigma, \\
u_n(0) = u_0^n & \text{in } \Omega.
\end{cases} \tag{1.12.13}$$

which is a consequence of the technique developed in [**Pe1**], based on the strong convergence of truncations $(T_k(u_n))$ with a simplify tools adapted from the "elliptic" idea of [**DMOP**].

LEMMA 1.48. *Let $\mu_s = \mu_s^+ - \mu_s^-$ be a bounded radon measure on Q, where μ_s^+ and μ_s^- are non-negative and concentrated, respectively, on two disjoint sets E^+ and E^- of zero $p-$capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \le \delta, \quad \mu_s^-(E^- \backslash K_\delta^-) \le \delta \tag{1.12.14}$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_0^1(Q)$, such that

$$\begin{aligned}
&\psi_\delta^+, \psi_\delta^- \equiv 1 \textit{ respectively on } K_\delta^+, K_\delta^-, \\
&0 \le \psi_\delta^+, \psi_\delta^- \le 1, \\
&Supp(\psi_\delta^+) \cap Supp(\psi_\delta^-) \equiv \emptyset.
\end{aligned} \tag{1.12.15}$$

Moreover

$$\|\psi_\delta^+\|_S \le \delta, \quad \|\psi_\delta^-\|_S \le \delta,$$

and in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\begin{aligned}
&\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \delta, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \le \delta, \\
&\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \delta, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \le \delta,
\end{aligned} \tag{1.12.16}$$

and both ψ_δ^+ and ψ_δ^- converge to zero weakly– in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes. Moreover, if $\lambda_n = \lambda_n^\oplus - \lambda_n^\ominus$ is as in (1.12.12), we have*

$$\begin{aligned}
&\int_Q \psi_\delta^- \lambda_n^\oplus = \omega(n,\delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \le \delta,\\
&\int_Q \psi_\delta^+ \lambda_n^\ominus = \omega(n,\delta), \quad \int_Q \psi_\delta^+ d\mu_s^- \le \delta,\\
&\int_Q (1-\psi_\delta^+) \lambda_n^\oplus = \omega(n,\delta), \quad \int_Q (1-\psi_\delta^+) d\mu_s^+ \le \delta,\\
&\int_Q (1-\psi_\delta^-) \lambda_n^\ominus = \omega(n,\delta), \quad \int_Q (1-\psi_\delta^-) d\mu_s^- \le \delta.
\end{aligned} \tag{1.12.17}$$

Proof. (Sketch of the proof) We follow the lines of [**DMOP**] and [**Pe1**]. We recall that μ^+ and μ^- are concentrated on two disjoint subsets E^+ and E^- whose $p-$capacity is zero. Moreover, since μ^+ and μ^- are Radon measures, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that

$$\mu^+(E^+ \backslash K_\delta^+) \le \delta, \quad \mu^-(E^- \backslash K_\delta^-) \le \delta.$$

Since $K_\delta^+ \cap K_\delta^- = \emptyset$, there exist two disjoint open subsets A_δ^+ and A_δ^- such that $K_\delta^+ \subseteq A_\delta^+$ (resp. $K_\delta^- \subseteq A_\delta^-$). Moreover, since $\text{cap}_p(K_\delta^+, Q) = 0$) (resp. $\text{cap}_p(K_\delta^-, Q) = 0$), we have that $\text{cap}_p(K_\delta^+, U_\delta^+) = 0$ (resp. $\text{cap}_p(K_\delta^-, U_\delta^-) = 0$) (see [**Pe1**], Lemma 4). Thus, by definition of parabolic $p-$capacity, there exist two functions $\varphi_\delta^+ \in C_0^\infty(U_\delta^+)$ (resp. $\varphi_\delta^- \in C_0^\infty(U_\delta^-)$) such that for every $\delta' > 0$,

$$\|\varphi_\delta^+\|_W \le \delta' \text{ and } \varphi_\delta^+ \ge \chi_{K_\delta^+} \quad (\text{resp. } \|\varphi_\delta^-\|_W \le \delta' \text{ and } \varphi_\delta^- \ge \chi_{K_\delta^-}).$$

Then we obtain (1.12.14) by taking $\psi_\delta^+ = \overline{H}(\varphi_\delta^+)$ (resp. $\psi_\delta^- = \overline{H}(\varphi_\delta^-)$) with ($H(s) = \frac{4}{3}$ if $|s| \le \frac{1}{2}$, 0 if $|s| > 1$, and affine if $\frac{1}{2} < |s| \le 1$)

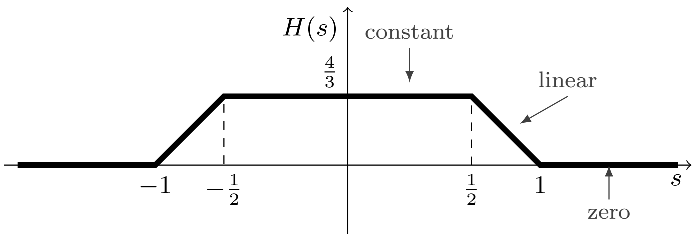


Figure 5. The function $H(s)$

Moreover, we have

$$\begin{aligned}
0 \le \int_Q \psi_\delta^- d\lambda^+ &= \int_{A_\delta^-} \psi_\delta^- d\lambda^+ \le \lambda^+(A_\delta^-) \le \lambda^+(Q \backslash A_\delta^+)\\
&\le \lambda^+(Q \backslash K_\delta^+) = \lambda^+(E^+ \backslash K_\delta^+) \le \delta
\end{aligned}$$

analogously

$$\int_Q \psi_\delta^+ d\lambda^- \le \delta.$$

Now let $\delta, \eta > 0$ fixed, we have

$$\begin{aligned}
0 \le \int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\lambda^+ &\le \int_{Q \backslash (K_\delta^+ \cap K_\eta^+)} (1 - \psi_\delta^+) d\lambda^+ \le \lambda^+(Q \backslash (K_\delta^+ \cap K_\eta^+))\\
&\le \lambda^+(Q \backslash K_\delta^+) + \lambda^+(Q \backslash K_\eta^+) \le \delta + \eta.
\end{aligned}$$

A similar result is obtained for the second inequality (1.12.16). □

Remark 1.49. If E^+ or (E^-) is closed (hence compact), we can choose $K_\delta^+ = E^+$ (K_δ^-) for $\delta > 0$. If for example $\lambda^+ = 0$, then we choose $K_\delta^+ = \emptyset$, and $\psi_\delta^+ \equiv 0$.

REMARK 1.50. Observe that as a consequence of (1.12.15), we have that both ψ_δ^+ and ψ_δ^- converge to zero as δ tends to zero, strongly in S^r, weakly–* in $L^\infty(Q)$ and almost everywhere in Q.

THEOREM 1.51. *Let u_n be solutions of* (1.12.13)*, with (μ_n) and (u_0^n) satisfying the assumptions* (1.12.12)*. Then there exists a positive constant C, not depending on n, and a positive constant C_k, which depends on k but not on n such that the following estimates hold true*

$$\begin{cases} \|u_n\|_{L^\infty(0,T;L^1(\Omega))} + \|u_n\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C, \\ \|T_k(u_n)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq Ck \quad \forall k > 0, \\ \|a(t,x,u_n,\nabla u_n)\|_{L^q(\Omega)^N} \leq C \quad \forall q < p - \dfrac{N}{N+1}, \\ \|a(t,x,T_k(u_n),\nabla T_k(u_n))\|_{L^{p'}(Q)^N} \leq C_k. \end{cases} \tag{1.12.18}$$

Moreover there exist a subsequence, still denoted by n, and a measurable function u belonging to the space $L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ such that $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$ and

$$\begin{cases} u_n \to u \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)), \text{ strongly in } L^1(Q) \text{ and a.e. in } Q, \\ T_k(u_n) \to T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q, \\ a(t,x,T_k(u_n),\nabla T_k(u_n)) \to \sigma_k \text{ wakly in } L^{p'}(Q)^N \text{ for every } k > 0, \\ a(t,x,u_n,\nabla u_n) \to \sigma \text{ weakly in } L^q(\Omega)^N \text{ for every } q < p - \dfrac{N}{N+1}, \end{cases} \tag{1.12.19}$$

where σ_k belongs to $L^{p'}(Q)^N$ and σ belongs to $L^q(\Omega)^N$ for every $q < p - \frac{N}{N+1}$.

PROOF. See [**BDGO, DO1, DPP**]. □

1.13. Duality solutions

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, $T > 0$, we denote by Q the cylinder $(0,T) \times \Omega$, we recall some properties of duality solutions in the case of linear operators with measures. Let consider the linear parabolic problems

$$\begin{cases} u_t - \operatorname{div}(M(t,x)\nabla u) = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x)) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \qquad \begin{cases} -w_t - \operatorname{div}(M^*(t,x)\nabla w) = g & \text{in } (0,T) \times \Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.13.1}$$

where M is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption

$$M(t,x)\ \zeta \cdot \zeta \geq \alpha |\zeta|^2 \quad \text{for any } \zeta \in \mathbb{R}^N \text{ and } \alpha > 0, \tag{1.13.2}$$

$M^*(t,x)$ is the transposed matrix of $M(t,x)$, $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q)$ the space of Radon measures with bounded total variation on Q.

In the elliptic case, the notion of duality solution of the Dirichlet problem was introduced in Section 1.5. Following the idea of Section 1.5, we can define a solution of (1.13.1) in a duality sense as follows

$$-\int_\Omega u_0 w(0)\ dx + \int_Q ug\ dxdt = \int_Q w\ d\mu, \tag{1.13.3}$$

for every $g \in L^\infty(Q)$, where w is the solution of the backward problem in (1.13.1). Note that all terms in (1.13.3) are well defined thanks to the standard parabolic regularity [**LSU**]. Moreover, it is quite easy to check that any duality solution of problem (1.13.1) actually turns out to be a distributional solution of the same problem. Finally a unique duality solution of problem (1.13.1) exists and we have the following result

THEOREM 1.52. *Let $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a unique duality solution of problem* (1.13.1).

Observe that by Theorem 1.52 a unique solution is well defined for all $t > 0$. Recall that in the case $\mu \in L^1(Q)$, u_0 smooth, $r, q \in \mathbb{R}$ fixed such that $r, q > 1$ and $\frac{N}{q} + \frac{2}{r} < 2$, for every $g \in L^r(0,T; L^q(\Omega)) \cap L^\infty(Q)$ and every duality solution w of the backward problem (1.13.1), standard results [**LSU**] implies that w is continuous on Q and

$$\|w\|_{L^\infty(Q)} \le C\|g\|_{L^r(0,T;L^q(\Omega))}.$$

Therefore, the linear functional $\Lambda : L^p(0,T; L^q(\Omega)) \to \mathbb{R}$ defined by $\Lambda(g) = \int_Q w \, d\mu + \int_\Omega u_0 w(0) \, dx$ is well defined and continuous, since

$$|\Lambda(g)| \le (\|\mu\|_{L^1(Q)} + \|u_0\|_{L^\infty(\Omega)})\|w\|_{L^\infty(Q)} \le C\|g\|_{L^r(0,T;L^q(\Omega))}.$$

So, by Riesz's representation theorem there exists a unique $u \in L^{r'}(0,T; L^{q'}(\Omega))$ such that $\Lambda(g) = \int_Q ug \, dxdt$ for any $g \in L^r(0,T; L^q(\Omega))$. So we have that, if $\mu \in L^1(Q)$ and u_0 is smooth, then there exists a (unique by construction) duality solution of problem (1.13.1). A standard approximation argument (see for instance Theorem 1.2 in [**BDGO**]) shows that a unique solution also exists for problem (1.13.1) if $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$.

Note that in the case where the measure μ does not depend on time, the duality solution which exists and is unique converges to the duality solution of the associated elliptic problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla v) = \mu & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\mathrm{div}(M^*(x)\nabla z) = g & \text{in } \Omega, \\ z(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13.4}$$

we recall that we mean by a duality solution of (1.13.4) a function $v \in L^1(\Omega)$ such that $\int_\Omega vg \, dxdt = \int_\Omega z \, d\mu$ for every $g \in L^\infty(\Omega)$, where z is the variational solution of the dual problem (1.13.4) (see Section 1.5). Then, a duality solution of (1.13.1) turns out to be continuous with values in $L^1(\Omega)$, and we have the following result

THEOREM 1.53. *Let $\mu \in \mathcal{M}_b(Q)$ be independent on the time t. Let $u(t,x)$ be the duality solution of problem* (1.13.1) *with $u_0 \in L^1(\Omega)$, and let $v(x)$ be the duality solution of the corresponding elliptic problem* (1.13.4). *Then $u(T,x)$ converges to $v(x)$ in $L^1(\Omega)$ as T tends to $+\infty$.*

1.14. Entropy solutions

We want to solve the parabolic equation

$$\begin{cases} u_t - \mathrm{div}(a(t,x,u,\nabla u)) = f & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.14.1}$$

with $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, $Q = (0,T) \times \Omega$ and Ω is an open bounded set of $\mathbb{R}^N$, a is a Carathéodory function satisfying coercivity, monotonicity and growth assumptions (1.11.1) $-$ (1.11.3) of Leray-Lions and defining an operator on $L^p(0,T; W_0^{1,p}(\Omega))$. Boccardo and Gallouët [**BG1**] have shown, in the more general case where $u_0 \in \mathcal{M}(\overline{\Omega})$ and $f \in \mathcal{M}([0,T] \times \overline{\Omega})$ (for $\mathcal{O}$ an open set $\mathcal{M}(\overline{O}) = (C(\overline{O}))'$ is the dual space of the space of continuous functions on $\overline{\Omega}$ endowed with its usual norm, it is a space of measures) that, if $p > 2 - \frac{1}{N+1}$, there exists a solution in the following sense

$$u \in \bigcap_{q < \frac{(p-1)(N+1)+1}{N+1}} L^q(0,T; W_0^{1,q}(\Omega))$$

and

$$-\int_0^T \int_\Omega u\varphi_t \, dxdt - \int_\Omega u_0 \varphi(0) \, dx + \int_0^T \int_\Omega a(t,x,\nabla u) \cdot \nabla\varphi \, dxdt = \int_0^T \int_\Omega f\varphi \, dx \tag{1.14.2}$$

for all $\varphi \in \mathcal{D}([0,T[\times\Omega)$. However this formulation does not ensure uniqueness for $N > 2$ as we see in Section 1.13 using the "elliptic" counter-example [**Pr1**] adapted from Serrin [**Ser**]. Indeed there exists a bounded and uniformly coercive matrix a, i.e.,

$$a_{ij}(x) \in L^\infty(\Omega), \quad \sum_{i,j} a_{ij} \, \zeta_i \cdot \zeta_j \ge \sum_i \zeta_i^2 \quad \forall \zeta_i \in \mathbb{R}^N \text{ a.e. } x \in \Omega,$$

and there exists $v(x) \neq 0$ such that $v \in \cap_{q<2-\epsilon} W_0^{1,q}(\Omega)$ (where $\epsilon > 0$ can be arbitrarily small, one chooses here $\epsilon < \frac{1}{N}$) that verifies

$$\begin{cases} -\mathrm{div}(A(x)\nabla v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of distributions. One sets $w(t,x) = v(x)$, then one has

$$\begin{cases} w_t - \mathrm{div}(A(t,x)\nabla w) = 0 & \text{in } (0,T) \times \Omega, \\ w(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ w(0,x) = v(x) & \text{in } \Omega, \end{cases}$$

in the sense of (1.14.2) with $w \in \cap_{q<\frac{N+2}{N+1}} L^q(0,T; W_0^{1,q}(\Omega))$ and $v \in L^2(\Omega)$ since $\epsilon < \frac{1}{N}$ **[Pr1]**. Since $v \in L^2(\Omega)$ there exists a variational solution $\tilde{w} \in L^2(0,T; H_0^1(\Omega))$ of the same problem **[LMa]**. Since $w \notin L^2(0,T; H_0^1(\Omega))$, because $v \notin H_0^1(\Omega)$, w and $\tilde{w}$ are not equal, thus $\overline{w} = w - \tilde{w} \neq 0$. Hence $\overline{w} \in \cap_{q<\frac{N+2}{N+1}} L^q(0,T; W_0^{1,q}(\Omega))$ is a solution, in the sense of distribution of (1.14.2), of

$$\begin{cases} \overline{w}_t - \mathrm{div}(A(t,x)\nabla \overline{w})) = 0 & \text{in } (0,T) \times \Omega, \\ \overline{w}(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ \overline{w}(0,x) = 0 & \text{in } \Omega, \end{cases}$$

with $\overline{w} \neq 0$. In order to obtain an existence and uniqueness result, an entropy formulation is proposed, it is very close to the one which has been introduced for the elliptic case in Section 1.7. In the case where $a(x,u,\nabla u)$ does not depend on t, existence and uniqueness of entropy solution have been proved, using semigroup theory in **[AMST]**, this formulation give a solution for problem (1.14.1).

Definition 1.54. For $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and Ω an open bounded set of $\mathbb{R}^N$, we define a entropy solution of (1.14.1) as a function $u \in C(0,T; L^1(\Omega))$ such that for all $k > 0$, $T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega))$ and

$$\begin{aligned} &\int_\Omega \Theta_k(u-\varphi)(T)\, dx - \int_\Omega \Theta_k(u_0 - \varphi(0))\, dx + \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle\, dt \\ &+ \int_0^T \int_\Omega a(t,x,u,\nabla u)\cdot \nabla T_k(u-\varphi)\, dxdt \leq \int_0^T \int_\Omega f T_k(u-\varphi)\, dxdt \end{aligned} \tag{1.14.3}$$

for all $k > 0$ and $\varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,T]; L^1(\Omega))$ such that $\varphi_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$. Then we have the following result

Theorem 1.55. *Let Ω be an open bounded set of $\mathbb{R}^N$, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and a satisfies* (1.11.1) − (1.11.3)*, then there exists one entropy solution of problem* (1.14.1).

We consider now the nonlinear equation (1.14.1) with the initial condition u_0 in $L^1(\Omega)$ and the right-hand side is a smooth measure μ on Q which is absolutely continuous with respect to the parabolic capacity associated with the operator $-\mathrm{div}(a(t,x,u,\nabla u))$. We extend the previous notion of entropy solution, which is generalization of Definition 1.54 given in **[Pr2]**. To this end, we define

$$E = \{\varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \text{ such that } \varphi_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)\}.$$

According to **[Po1]**, one has $E \subset C([0,T]; L^1(\Omega))$.

Definition 1.56. Under hypothesis (1.11.1) − (1.11.3), if $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_0(Q)$ and $(f, -\mathrm{div}(G), g)$ is a decomposition of μ according to Lemma 1.43, an entropy solution of (1.14.1) is a measurable function u such that

$$T_k(u-g) \in L^p(0,T; W_0^{1,p}(\Omega)) \ \text{ for all } k \geq 0, \tag{1.14.4}$$

$$\begin{aligned} &t \in [0,T] \mapsto \int_\Omega \Theta_k(u-g-\varphi)(t,x)\, dx \text{ is (a.e. equal to) a continuous function,} \\ &\text{for all } k \geq 0 \text{ and all } \varphi \in E, \end{aligned} \tag{1.14.5}$$

$$
\begin{aligned}
&\int_\Omega \Theta_k(u-g-\varphi)(T,x)dx - \int_\Omega \Theta_k(u_0(x)-\varphi(0,x))dx + \int_0^T \langle \varphi_t, T_k(u-g-\varphi)\rangle dt \\
&+ \int_Q a(t,x,u,\nabla u)\cdot\nabla(T_k(u-g-\varphi))dxdt \le \int_Q fT_k(u-g-\varphi)dxdt \\
&+ \int_Q G\cdot\nabla(T_k(u-g-\varphi))dxdt, \quad \text{for all } k\ge 0 \text{ and all } \varphi\in E.
\end{aligned}
\tag{1.14.6}
$$

Remark that in (1.14.6), we denote by $\langle\cdot,\cdot\rangle$ the duality product between $W^{-1,p'}(\Omega)+L^1(\Omega)$ and $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and the definition chosen of entropy solution in (1.14.6) uses an inequality instead of an equality, this a standard choice for entropy solutions because it's sufficient to obtain the uniqueness (in the case when a does not depend on u and $\mu\in L^1(Q)$ for example, see [**Pr2**]) and makes the proof of the existence quite easier (there is no need to prove the strong convergence of gradient of the approximate solutions).

1.15. Renormalized solutions

In the section 1.8, we have seen that under assumptions $(a1)-(a3)$ there exist a renormalized solutions of elliptic problem (1.8.1) and we extended the results of existence of a weak solution to the case of bounded Radon measure on Ω. this is due to the approximations in the energy space $W_0^{1,p}(\Omega)$. One may wonder what happens in the evolution case. In this case, if $f\in L^1(Q)$ and $u_0\in L^1(\Omega)$, the existence of a weak solution has been proved in [**Po1**] if $p=2$, this solution does not have finite energy and it only belongs to $L^q(0,T;W_0^{1,q}(\Omega))$ for any $q<\frac{N+2}{N+1}$. For completeness, let us then rewrite the setting of our assumptions and let us consider the initial boundary value problem

$$
\begin{cases}
u_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu & \text{in } Q,\\
u(t,x)=0 & \text{on } \Sigma,\\
u(0,x)=u_0(x) & \text{in } \Omega,
\end{cases}
\tag{1.15.1}
$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N\ge 1$, $T>0$ and Q is the cylinder $\Omega\times(0,T)$, Σ being its lateral surface. We assume that

$$\mu\in\mathcal{M}_b(Q),\ \ u_0\in L^1(\Omega).$$

The main point in our study, as in the elliptic case, is the relationship between the possibility to find solutions of (1.15.1) and the stability properties of the equation, as they naturally arise when one tries to solve (1.15.1) by a density argument, that is approximating the singular data μ and u_0 with sequences of smooth functions. For example, letting (μ_n) and u_0^n be a standard approximation of μ and u_0 constructed by convolution, we need to study the behaviour of the sequence (u_n) of solutions of the following problems

$$
\begin{cases}
u_t - \operatorname{div}(a(t,x,u_n,\nabla u_n)) = \mu_n & \text{in } Q,\\
u_n(t,x)=0 & \text{on } \Sigma,\\
u_n(0,x)=u_0^n(x) & \text{in } \Omega.
\end{cases}
\tag{1.15.2}
$$

The stability properties proved on the solutions of (1.15.2) lead us to the problem of finding a suitable definition of solution of (1.15.1) which may provide existence and stability at the same time. The comparison with the results of the stationary case suggests that a good notion of solution which satisfies these requirements is the notion of renormalized solution. This is why we choose to carry on the whole study in this framework, proving the existence result directly through the proof of the stability of renormalized solutions, which also includes the study of (1.15.2) as n tends to ∞, since for smooth data renormalized solutions and weak solutions coincide. Once more, we recall that renormalized solutions were introduced in [**DL1, DL2**] to deal with first order hyperbolic equations of conservation laws. This notion was then developed for parabolic problems in [**BM**], in several papers afterwards (see the references in [**BMR**]), and in case of L^1-data in order to get uniqueness of solutions. Following the ideas of [**DMOP**] for the stationary problem, we provide here the definition of renormalized solution for the initial boundary value problem (1.15.1) with measures as data. In the spirit of [**DPP**], we give this definition for a soft measures (absolutely continuous measures with respect to capacity) and then we say that this notion always yield for general measures. Henceforward we consider, for every μ_0 in

$\mathcal{M}_0(Q)$, its absolutely continuous decomposition by writing

$$\mu_0 = f - \operatorname{div}(G) + g_t, \quad \mu_0(E) = 0 \text{ with } \operatorname{cap}_p(E) = 0.$$

Moreover, we will always denote by $C_c^\infty([0,T] \times \Omega)$ the set of functions φ in $C^\infty(\overline{Q})$ such that $\varphi = 0$ in $\Sigma \cup \{\Omega \times \{T\}\}$.

DEFINITION 1.57. A measurable function u is said to be a renormalized solution of (1.15.1) if

$$u - g \in L^\infty(0,T;L^2(\Omega)),\ T_k(u-g) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k > 0, \tag{1.15.3}$$

$$\lim_{n\to\infty} \int_{\{(t,x):\ n \le u \le n+1\}} a(t,x,u,\nabla u) \cdot \nabla u\ \varphi\ dxdt = 0 \quad \forall \varphi \in C(\overline{Q}), \tag{1.15.4}$$

for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support, u satisfies in the sense of distributions in Q

$$\begin{aligned}&(S(u-g))_t - \operatorname{div}\big(a(t,x,u,\nabla u)S'(u-g)\big) + a(t,x,u,\nabla u)\cdot\nabla u S''(u-g)\\ &= S'(u-g)f + S''(u-g)G\cdot\nabla(u-g) - \operatorname{div}\big(GS''(u-g)\big)\end{aligned} \tag{1.15.5}$$

and

$$S(u-g)(0) = S(u_0) \text{ in } L^1(\Omega). \tag{1.15.6}$$

Let us remark that the renormalized formulation is obtained, as usual, through the formal multiplication of equation (1.15.1) by $S'(u)$ where S belongs to $W^{2,\infty}(\mathbb{R})$ and S' has compact support. Then all the terms in (1.15.5) have a meaning since $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. Let us also remark that (1.15.5) can be asked to hold in the weaker sense of distributions, that is

$$\begin{aligned}&-\int_0^T S(u-g)\psi_t dt - \int_0^T \psi\ \operatorname{div}(a(t,x,\nabla u)S'(u-g))dt + \int_0^T \psi\ a(t,x,\nabla u)\cdot\nabla(u-g)S''(u-g)dt\\ &= \int_0^T \psi S'(u-g)d\mu \text{ in } \mathcal{D}'(\Omega).\end{aligned}$$

Then since $S(u-g)_t$ belongs to $L^p(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, using a density result we recover that (1.15.5) holds in the sense of distributions in Q. Note that the renormalized solution is also a weak solution. It is also easy to prove that the two concepts are in fact equivalent if μ and u_0 belong respectively to $L^p(0,T;W^{-1,p'}(\Omega))$ and $L^2(\Omega)$.

PROPOSITION 1.58. *Every renormalized solution is a weak solution, the reverse being true if μ belongs to $L^p(0,T;W^{-1,p'}(\Omega))$ and u_0 is in $L^2(\Omega)$.*

PROOF. The proof is trivial, we can just multiply by $S(u-g)\varphi$ and let S to 1. □

To investigate the stability properties of renormalized solutions, which also include as a consequence of Proposition 1.58, the stability of the behaviour, as n tends to infinity, of the approximating sequence (u_n) of solutions of (1.15.2) where μ_n converges tightly to μ and u_0^n converges weakly to u_0 in $L^1(Q)$, that is,

$$\lim_{n\to\infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu_0 \quad \forall \varphi \in C(\overline{Q}). \tag{1.15.7}$$

Under the assumptions (1.11.1) − (1.11.3), the stability properties of the renormalized solutions with respect to the data (u_0^n, μ_n) are strongly related to the compactness of the sequence $T_k(u_n)$ in the strong topology of the energy space $L^p(0,T;W_0^{1,p}(\Omega))$. This kind of compactness result on the truncations of solutions plays a crucial role in the existence theory for nonlinear equations with integrable or measure data. As for parabolic initial boundary value problems, the strong convergence in $L^p(0,T;W_0^{1,p}(\Omega))$ of truncations of solutions of approximating problems was proved, in case of L^1 data, in [**B**] (see also [**BMR**]).

THEOREM 1.59. *Let $\mu_n \subset \mathcal{M}_0(Q)$ be a sequence of measures tightly converging to μ in $\mathcal{M}_b(Q)$ and let u_0^n weakly converges to u_0 satisfying* (1.12.12). *Let u_n be renormalized solutions of* (1.15.2) *in the sense of Definition 1.57. Then there exist a measurable function u, and a subsequence u_n, such that*

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k > 0.$$

PROOF. See [**DPP**], Proposition 3.14 □

THEOREM 1.60. *Let $\mu \in \mathcal{M}_b(Q)$ and let $u_0 \in L^1(Q)$. Then there exists a unique renormalized solution u of* (1.15.1). *Moreover u satisfies the additional regularity $u \in L^\infty(0,T;L^1(\Omega))$ and $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$.*

PROOF. See [**DPP**], Theorem 1.3. □

Notice that the notion of renormalized solution and entropy solution for parabolic problem (1.15.1) turn out to be equivalent as proved in [**DP**], in Chapters 4, 6 and 8 we extend this notion of renormalized solution for general measure data $\mu \in \mathcal{M}_b(Q)$ and so, thanks to this result, this notion will turn out to be coherent with all definitions of solution given before for problem (1.15.1).

1.16. Parabolic equations with absorption term

Let Ω be a bounded domain of $\mathbb{R}^N$ and $Q = \Omega \times (0,T)$, we consider perturbed problems of the model type

$$
\begin{cases} u_t - \operatorname{div}(a(t,x,u,\nabla u)) + G(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.16.1}
$$

where $G(u)$ may be an absorption or source term. In the model case $G(u) = \pm|u|^{q-1}u$ with $q > p-1$, or G has an exponential type. We give existence results when q is subcritical, or when the measure μ is good in time and satisfies suitable capacity conditions. First we consider the case of an absorption term $G(u)u \geq 0$.

Let us recall the case $p = 2$, $a(t,x,u,\nabla u) = \nabla u$ and $G(u) = |u|^{q-1}u$ with $q > 1$. The first results concern the case $\mu = 0$ and u_0 is a Dirac mass in Ω, see [**BF**], existence holds if and only if $q < \frac{N+2}{N}$. Then optimal results are given in [**BPi1**] for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $C_{\alpha,k}$ with $\alpha, k > 1$ defined, for any Borel set $E \subset \mathbb{R}^N$, by

$$
C_{\alpha,k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N),\ G_\alpha * \varphi \geq \chi_E\}, \tag{1.16.2}
$$

where G_α is the Bessel kernel of order α, and a capacity $C_{G,k}$ with $k > 1$ adapted to the operator of the heat equation of kernel $G(x,t) = \chi_{(0,\infty)}(4\pi t)^{-\frac{N}{2}} e^{\frac{-|x|^2}{4t}}$, for any Borel set $E \subset \mathbb{R}^{N+1}$, by

$$
C_{G,k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}),\ G * \varphi \geq \chi_E\}. \tag{1.16.3}
$$

From [**BPi1**], there exists a solution if and only if μ does not charge the sets of $C_{G,q'}(E)$ capacity zero and u_0 does not charge the sets of $C_{\frac{2}{q},q'}$ $-$capacity zero. Observe that one can reduce to a zero initial data, by considering the measure $\mu + u_0 \otimes \delta_0^t$ in $\Omega \times (-T,T)$, where $\otimes$ is the tensor product and δ_0^t is the Dirac mass in time at 0.

For $p \neq 2$ such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case $\mu = 0$ with Ω bounded, or $\Omega = \mathbb{R}^N$. The case where u_0 is a Dirac mass in Ω is studied in [**Gm**], [**KV**] when $p > 2$, and [**CQW**] when $p < 2$. Existence and uniqueness hold in the subcritical case $q < p-1+\frac{p}{N}$. If $q \geq p-1+\frac{p}{N}$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p-1+\frac{p}{N}$, and $u_0 \in \mathcal{M}_b^+(\Omega)$, the existence is obtained in the sense of distributions in [**Zh**], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [**BCV**]. The case $\mu \in L^1(Q)$, $u_0 = 0$ is treated in [**DO1**], and $\mu \in L^1(Q)$, $u_0 \in L^1(\Omega)$ in [**ASW**] where G can be multivalued. The case $\mu \in \mathcal{M}_0(Q)$ is studied in [**PPP2**] with a new formulation of the solutions, and existence and uniqueness are obtained for any function $G \in C(\mathbb{R})$ such that $G(u)u \geq 0$. Up to now an existence result have been obtained for a general measure $\mu \in \mathcal{M}_b(Q)$ with measures concentrated on a set of zero parabolic $r-$capacity with $1 < p < r$ and q large enough in [**Pe2**]. The case of a source term $G(u) = -u^q$ with $u \geq 0$ has been treated in [**BPi2**] for $p = 2$, where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general p.

In order to deal with all problems mentioned above, we will often make use some auxiliary functions (already used in the stationary problems) linking the entropy, renormalized and weak formulations. Being

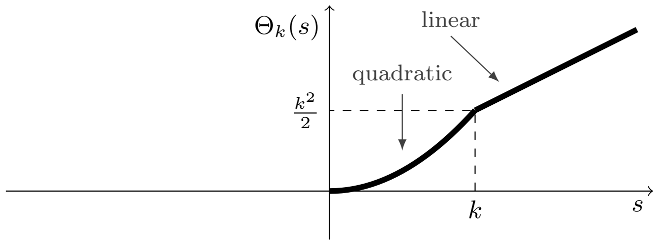


FIGURE 6. The function $\Theta_k(s)$

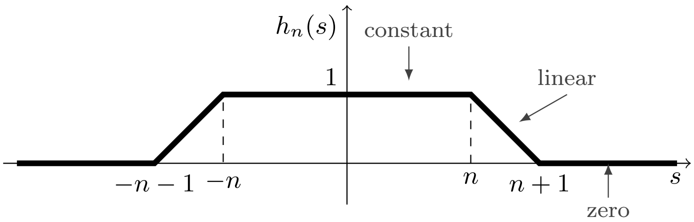


FIGURE 7. The function $h_n(s)$

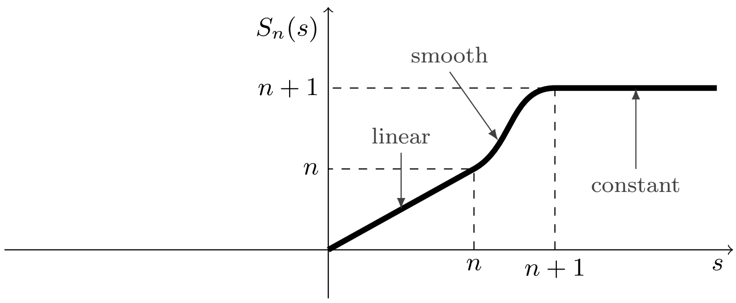


FIGURE 8. The function $S_n(s)$

$T_k(s)$ the truncation function at levels $\pm k$ defined in Figure 1, we define

$$\Theta_k(s) = \int_0^s T_k(\tau)d\tau, \quad \forall s \in \mathbb{R}.$$

$$h_n(s) = 1 - |T_1(s - T_n(s))|, \quad \forall s \in \mathbb{R}.$$

In particular, we note that $h_n(s)$ is such that $\mathrm{Supp}(h_n) = [-n-1, n+1]$ (i.e., has a compact support), and that $h_n(s)$ tends to 1 as n tends to infinity for every $s \in \mathbb{R}$. So that, the functions $S_n(s)$ are defined by

$$S_n(s) = \int_0^s h_n(r)dr, \quad \forall s \in \mathbb{R}.$$

Moreover, S_n converges as n tends to infinity, to the identity function $I(s) = s$.

Finally, let us make clear some notations that will be used in the remaining of this Chapters are introduced as well as in Section 1.1. We will often introduce in our proof different parameters, such as δ which tends to zero, or k which tends to infinity. Then we will denote by those terms such that: If a quantity does not depend on one of the parameters we will omit to write this one in the notation, writing for instance $\omega(n, \delta)$ for a term which does not depend on k at all. On the other hand, we will use the notation $\omega(k, \delta(n))$ to denote a term which converges to zero as n tends to infinity for every fixed δ and k. In fact, the order in which the parameters converge is essential in what follows.

1.17. Variable exponent Lebesgue–Sobolev spaces

We recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. We refer to X. Fan and D. Zhao **[FZ1, FZ2]** for further properties on variable exponent Lebesgue-Sobolev spaces. We start with a brief overview of the state of the art concerning elliptic spaces with variable exponent and parabolic spaces modeled upon them. Another area where these spaces have found applications is the study of electrorheological fluids, see the papers by Diening alone [**Die2**] and with Růžička [**DR**] on the role of variable exponent in this context. The same spaces appear also in the study of variational integrals with non-standard growth, see [**AM, CN, Zhi**]. First of all, let us introduce the following notations

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and given a bounded measurable function $p(\cdot) : \Omega \to \mathbb{R}$, the critical Sobolev exponent and the conjugate of $p(\cdot)$ are respectively

$$p^\star(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)} \quad \text{and} \quad p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

We define the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular $\rho_{p(\cdot)}(\Omega) = \int_\Omega |u|^{p(x)}dx$ is finite, i.e.,

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R},\ u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

If the exponent is bounded, i.e., if $p_+ < \infty$, we define a norm in $L^{p(\cdot)}(\Omega)$, called *the Luxembourg norm*, by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0,\ \rho_{p(\cdot)}(\frac{u}{\lambda})dx = \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\}.$$

The following inequality will be used later

$$\min\{\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}\} \le \int_\Omega |u(x)|^{p(x)}dx \le \max\{\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}\}. \tag{1.17.1}$$

The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}})$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the following *Hölder's inequality*

$$\int_\Omega |uv|dx \le (\frac{1}{p_-} + \frac{1}{p'_-})\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \quad \forall u \in L^{p(\cdot)}(\Omega),\ \forall v \in L^{p'(\cdot)}(\Omega) \tag{1.17.2}$$

holds true. One central property of $L^{p(\cdot)}(\Omega)$ is that the norm and the modular topology coincide, i.e., $\rho_{p(\cdot)}(u_n) \to 0$ if and only if $\|u_n\|_{L^{p(\cdot)}} \to 0$. We define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega),\ |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with one of the following equivalent norms

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

or

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf\{\lambda > 0,\ \int_\Omega (|\frac{\nabla u(x)}{\lambda}|^{p(x)} + |\frac{u(x)}{\lambda}|^{p(x)})dx \le 1\}.$$

By $W_0^{1,p(\cdot)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

Assuming $p^- > 1$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces and the space $W^{-1,p'(\cdot)}(\Omega)$ denotes the dual of $W_0^{1,p(\cdot)}(\Omega)$.

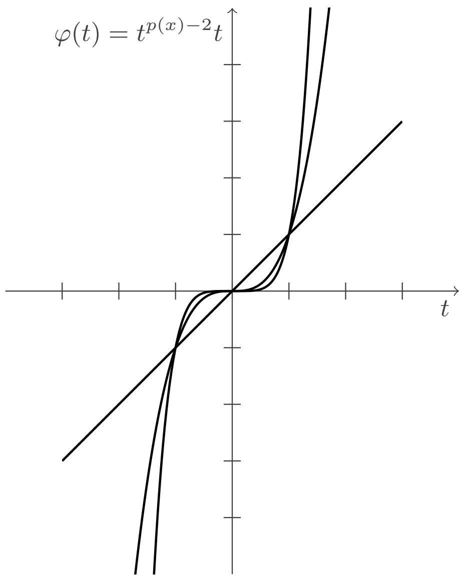


FIGURE 9. The function $t^{p(x)-2}t$ for $p(x) = 2, 4, 6$

The following condition has emerged as the right one to guarantee regularity of variable exponent Lebesgue spaces. We say that $p(\cdot)$ is Log-Hölder continuous if $p(\cdot) : \Omega \to \mathbb{R}$ is a measurable function such that

$$\begin{aligned} &\exists C > 0 : \quad |p(x) - p(y)| \leq \frac{C}{-ln|x-y|}, \quad \text{for} \quad |x-y| < \frac{1}{2}, \\ &1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x) < N. \end{aligned} \tag{1.17.3}$$

This condition is also called Dini-Lipschitz, weak-Lipschitz and $0-$Hölder condition. The Log-Hölder continuity condition is used to obtain several regularity results for Sobolev spaces with variable exponents, in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $p^- \geq 1$, the Poincaré inequality holds **[HHKV2]** for some constant C which depends on Ω and the function $p(\cdot)$. The proofs of the following Propositions can be found in **[FZ1, KR, FSZ]**, respectively (see **[Die1]** for more details).

PROPOSITION 1.61 (The $p(\cdot)-$Poincaré inequality). *Let Ω be a bounded open set and let $p(\cdot) : \Omega \to [1, \infty)$ satisfy* (1.17.3). *Then there exists a constant C, depending only on $p(\cdot)$ and Ω, such that the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{1.17.4}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$.

Note that the following inequality $\int_\Omega |u|^{p(x)}dx \leq C \int_\Omega |\nabla u|^{p(x)}dx$, in general does not hold **[FZ1]**.

PROPOSITION 1.62 (Sobolev embedding 1). *Let Ω be a bounded open set, with a Lipschitz boundary, and let $p(\cdot) : \Omega \to [1, \infty)$ satisfy* (1.17.3). *Then we have the following continuous embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega), \ \textit{with}\ p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)}.$$

PROPOSITION 1.63 (Sobolev embedding 2). *For $p(\cdot) \in C(\overline{\Omega})$ with $1 < p^- \leq p^+ < N$, the Sobolev embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega), \tag{1.17.5}$$

hold, for every measurable function $r(\cdot) : \Omega \to [1, +\infty)$ such that $\operatorname*{ess\,inf}_{x\in\Omega} (\frac{Np(x)}{N-p(x)} - r(x)) > 0$.

For $T > 0$ let $Q := (0, T) \times \Omega$. Extending the variable exponent $p(\cdot) : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q}$, we may also consider the generalized Lebesgue space (which, of course, shares the same type of properties as $L^{p(\cdot)}(\Omega)$)

$$L^{p(\cdot)}(Q) = \{u : Q \to \mathbb{R},\ u \text{ is measurable with } \int_Q |u(t,x)|^{p(x)}\, dxdt < \infty\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)} = \inf\{\lambda > 0,\ \int_0^T \int_\Omega |\frac{u(t,x)}{\lambda}|^{p(x)} dxdt \le 1\}.$$

Moreover, if $p(\cdot)$ is log-Hölder continuous in Ω, so it is in Q . Indeed, if $p(\cdot)$ satisfies the log-Hölder continuity condition in Ω, according to (1.17.3), there exists a non-decreasing function $\omega : (0, \infty) \to \mathbb{R}$ such that $\limsup_{t\to 0^+} \omega(t) ln(\frac{1}{t}) < +\infty$ and

$$|p(t,x) - p(s,y)| = |p(x) - p(y)| < \omega(|x - y|) \le \omega(|(t,x) - (s,y)|),$$

holds for all $(t, x), (s, y) \in \overline{Q}$ such that $|(t, x) - (s, y)| < 1$.

If V is a Banach space, We will also use the standard notations for Bochner spaces, if $1 \le q \le \infty$ and $T > 0$, then $L^q(0, T; V)$ denotes the space of strongly measurable functions $u : (0, T) \to V$ such that $t \to \|u(t)\|_V \in L^q(0, T)$. Moreover, $\mathcal{C}([0, T]; V)$ denotes the space of continuous functions $u : [0, T] \to V$ endowed with the norm $\|u\|_{C([0,T];V} = \max_{t\in[0,T]} \|u(t)\|_V$. The following density result will be used in the study of the evolution problems.

Proposition 1.64. *Let $V = L^p(\Omega)$ or $V = W^{1,p}(\Omega)$ and $1 \le p < \infty$. Then, $\mathcal{D}((0, T) \times \Omega)$ is dense in $L^q(0, T; V)$ for any $1 \le q < \infty$.*

Proof. From **[Dr]**, Corollary 1.3.1, it follows that

$$Z := \left\{ \sum_{i=1}^n \phi_i(x)\psi_i(t),\ n \ge 1,\ \phi_i \in \mathcal{D}(\Omega),\ \psi_i \in \mathcal{D}(0,T) \right\} \subset \mathcal{D}((0,T) \times \Omega)$$

is dense in $L^q(0, T; V)$ for any Banach space V such that $\mathcal{D}(\Omega)$ is dense in V and $1 \le q < \infty$. □

Let $p(\cdot) : \overline{\Omega} \to [1, \infty)$ be a continuous variable exponent and $T > 0$. The abstract Bochner spaces $L^{p^+}(0, T; L^{p(\cdot)}(\Omega))$ and $L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ will be important in the study of parabolic problems. In the following we identify an abstract function like $v \in L^{p_-}(0, T; L^{p(\cdot)}(\Omega))$ with the real-valued function v defined by $v(t, x) = v(t)(x)$ for almost all $t \in (0, T)$ and almost all $x \in \Omega$. In the same way we associate to any function $v \in L^{p(\cdot)}(Q)$ an abstract function $v : (0, T) \to L^{p(\cdot)}(\Omega)$ by setting $v(t) := v(t, \cdot)$ for almost every $t \in (0, T)$.

Lemma 1.65. *We have the following continuous dense embeddings*

$$L^{p_+}(0, T; L^{p(\cdot)}(\Omega)) \overset{d}{\hookrightarrow} L^{p(\cdot)}(Q) \overset{d}{\hookrightarrow} L^{p_-}(0, T; L^{p(\cdot)}(\Omega)). \tag{1.17.6}$$

Proof. For $v \in L^{p(\cdot)}(Q)$, the corresponding abstract function $v : (0, T) \to L^{p(\cdot)}(\Omega)$ is strongly Bochner measurable (by the Dunford-Pettis Theorem, since it is weakly measurable and $L^{p(\cdot)}(\Omega)$ is separable). Moreover, using

$$\begin{aligned}
\int_0^T \|v(t)\|^{p_-}_{L^{p(\cdot)}(\Omega)} dt &\le \int_0^T \max\left[\int_\Omega |v(t,x)|^{p(x)}dx,\ (\int_\Omega |v(t,x)|^{p(x)}dx)^{\frac{p_-}{p_+}}\right] dt \\
&\le \int_0^T \int_\Omega |v(t,x)|^{p(x)} dxdt + T^{1-\frac{p_-}{p_+}} \left(\int_0^T \int_\Omega |v(t,x)|^{p(x)} dxdt\right)^{\frac{p_-}{p_+}} \\
&\le \max\left[|v|^{p_-}_{L^{p(\cdot)}(Q)}, |v|^{p_+}_{L^{p(\cdot)}(Q)}\right] + T^{1-\frac{p_-}{p_+}} \max\left[\|v\|^{\frac{(p_-)^2}{p_+}}_{L^{p(\cdot)}(Q)}, \|v\|^{p_-}_{L^{p(\cdot)}(Q)}\right].
\end{aligned} \tag{1.17.7}$$

Therefore, the embedding of $L^{p(\cdot)}(Q)$ into $L^{p_-}(0,T;L^{p(\cdot)}(\Omega))$ is continuous. If $u \in L^{p_+}(0,T;L^{p(\cdot)}(\Omega))$, from $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ it follows that $u \in L^{p_+}(0,T;L^1(\Omega))$, hence, according to **[Dr]**, Proposition 1.8.1, the corresponding real-valued function $u : (0,T) \times \Omega \to \mathbb{R}$ is measurable and using the same arguments as above we find the continuous embedding of $L^{p_+}(0,T;L^{p(\cdot)}(\Omega))$ into $L^{p(\cdot)}(Q)$. It is left to prove that both embeddings are dense. We consider the first embedding and fix $u \in L^{p(\cdot)}(Q)$. Since $\mathcal{D}(Q)$ is dense $L^{p(\cdot)}(Q)$, we find a sequence $(u_n) \subset \mathcal{D}(Q)$ converging to u in $L^{p(\cdot)}(Q)$ as $n \to \infty$. According to Proposition 1.64, $\mathcal{D}(Q)$ is densely embedded into $L^{p_+}(0,T;L^{p_+}(\Omega))$, therefore $u_n \in L^{p_+}(0,T;L^{p(\cdot)}(\Omega))$ for all $n \in \mathbb{N}$. To prove the denseness of the second embedding, we fix $v \in L^{p_-}(0,T;L^{p(\cdot)}(\Omega))$. Taking a standard sequence of mollifiers $(\rho_n)_n \subset \mathcal{D}(\mathbb{R})$ and extending v by zero onto $\mathbb{R}$, from **[Dr]**, Proposition 1.7.1, it follows that the regularized (in time) function

$$(\rho_n * v)(\cdot) := \int_{\mathbb{R}} \rho_n(\cdot - s) v(s) ds \tag{1.17.8}$$

is in $L^{p_+}(\mathbb{R}, L^{p(\cdot)}(\Omega))$ for each $n \in \mathbb{N}$, hence in $L^{p(\cdot)}(Q)$ and converges to v in $L^{p_-}(0,T;L^{p(\cdot)}(\Omega))$ (see **[Dr]**, Theorem 1.7.1). □

1.18. Orlicz-Sobolev spaces

In this final Section we present some results involving replacement of the spaces $L^p(\Omega)$ with more general spaces $L_A(\Omega)$ in which the role usually played by the convex function t^p is assumed by more general convex functions $A(t)$. The spaces $L_A(\Omega)$, called *Orlicz spaces* are studied in depth in the monograph by Krasnosel'skii and Rutickii **[KrR]** and also in the doctoral thesis by Luxemburg **[Lux]**. For a more complete developments we refer to the books by Adams **[A]**, Adams with Hedberg **[AH]**, Musielak **[Mus]**, to the Monograph of Rao with Ren **[RR]**, and to the papers by Gossez **[G1, G2, G3]**, Gossez and Benkirane **[BGo]**, Benkirane and Elmahi **[BEl1, BEl2]** and Elmahi **[El]**. Following Krasnosel'skii and Rutickii **[KrR]**, we use the class of "$N-$functions" as defining functions A for Orlicz spaces, this class is not as wide as the claas of Young's functions used by Luxemburg **[Lux]** (see also O'Neill **[O]**), for instance, it excludes $L^1(\Omega)$ and $L^\infty(\Omega)$ from the class of Orlicz spaces. However, $N-$functions are simpler to deal with and are adequate for our purposes. If necessary, it's possible to use more general Young's functions.

If the role played by $L^p(\Omega)$ in the definition of the Sobolev space $W^{m,p}(\Omega)$ is assigned instead to an Orlicz space $L_A(\Omega)$, the resulting space is denoted by $W^m L_A(\Omega)$ and called *Orlicz-Sobolv* spaces. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Donaldson and Trudinger **[DT]**. Let a be a real valued function defined on $[0,\infty)$ and having the following properties

(a) $\quad a(0) = 0,\ a(t) > 0$ if $t > 0,\ \underset{t\to\infty}{a(t)} = \infty,$

(b) $\quad a$ is nondecreasing, that is, $s > t \geq 0$ implies $a(s) \geq a(t)$,

(c) $\quad a$ is right continuous, that is, if $t \geq 0,$ then $\lim_{s \to t^+} a(s) = s(t).$

Then the real valued function A defined on $[0,\infty)$ by

$$A(t) = \int_0^t a(\tau) d\tau, \tag{1.18.1}$$

is called an *$N-$function.* It is not difficult to verify that any such $N-$function A has the following properties

(i) $\quad A$ is continuous on $[0,\infty)$,

(ii) $\quad A$ is strictly increasing, that is, $s > t \geq 0$ implies $A(s) > A(t)$,

(iii) $\quad A$ is convex, that is, if $s,t \geq 0$ and $0 < \lambda < 1,$ then $A(\lambda s + (1-\lambda)t) \leq \lambda A(s) + (1-\lambda)A(t)$,

(iv) $\quad \lim_{t\to 0^+} \dfrac{A(t)}{t} = 0,\ \lim_{t\to\infty} \dfrac{A(t)}{t} = \infty,$

(v) $\quad$ if $s > t > 0,$ then $\dfrac{A(s)}{s} > \dfrac{A(t)}{t}.$

Properties (i), (ii), (iv) could have been used to define $N-$function since they imply the existence of a representation of A in the form (1.17.8) with a having the required properties $(a) - (c)$. The following are examples of $N-$functions

$$A(t) = t^p,\ 1 < p < \infty,$$
$$A(t) = e^t - t - 1,$$
$$A(t) = e^{t^p} - 1,\ 1 < p < \infty,$$
$$A(t) = (1+t)\log(1+t) - t.$$

Evidently $A(t)$ is represented by the area under the graph $\sigma = a(\tau)$ from $\tau = 0$ to $\tau = t$ as shown in Figure 10. Rectilinear segments in the graph of A correspond to intervals of constancy of a, and angular points in the graph of A correspond to discontinuities (i.e., vertical jumps) in the graph of a

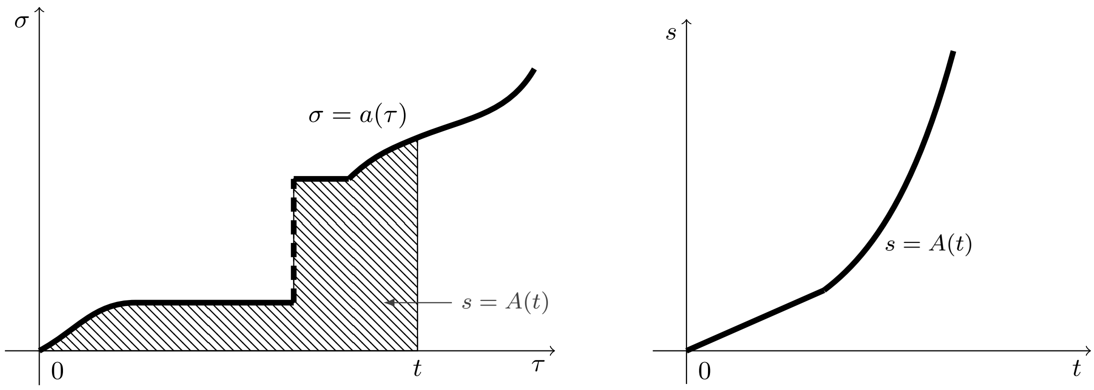


FIGURE 10. The functions $a(\tau)$ and $A(t)$

Given a satisfying $(a) - (c)$, we define

$$\tilde{a}(s) = \sup_{a(t) \leq s} t. \tag{1.18.2}$$

It is readily checked that the function a so defined also satisfies $(a) - (c)$ and that a can be recovered from $\tilde{a}$ via

$$a(t) = \sup_{\tilde{a}(s) \leq t} s. \tag{1.18.3}$$

(if a is strictly increasing, then $\tilde{a} = a^{-1}$). The $N-$function A and $\tilde{A}$ given by

$$A(t) = \int_0^t a(\tau)d\tau, \quad \tilde{A}(s) = \int_0^s \tilde{a}(\sigma)d\sigma \tag{1.18.4}$$

are said to be *complementary*, each is the complement of the other. Examples of such complementary pairs are

$$A(t) = \frac{t^p}{p}, \quad \tilde{A}(s) = \frac{s^{p'}}{p'}, \quad 1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1,$$
$$A(t) = e^t - t - 1, \quad \tilde{A}(s) = (1+s)\log(1+s) - s.$$

$\tilde{A}$ is represented by the area to the left of the graph $\sigma = a(\tau)$ (or more correctly $\tau = \tilde{a}(\sigma)$) from $\sigma = 0$ to $\sigma = s$ as shown in Figure 11. Evidently we have

$$st \leq A(t) + \tilde{A}(s), \tag{1.18.5}$$

which is know as *Young's inequality.* Equality holds in (1.18.5) if and only if either $t = \tilde{a}(s)$ or $s = a(t)$. Writing (1.18.5) in the form

$$\tilde{A}(s) \geq st - A(t),$$

and noting that equality occurs when $a = \tilde{a}(s)$, we have

$$\tilde{A}(s) = \max_{t \geq 0}\ (st - A(t)).$$

The relationship could have been used as the definition of the $N-$function $\tilde{A}$ complementary to A

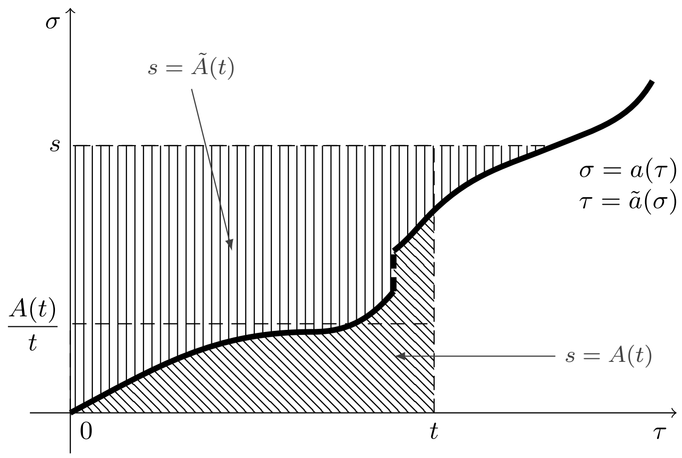


FIGURE 11. The functions $A(t)$ and $\tilde{A}(t)$

An $N-$function A is said to satisfy the global Δ_2-*condition* if there exists a positive constant k such that for every $t \geq 0$,

$$A(2t) \leq kA(t). \tag{1.18.6}$$

Similarly A is said to satisfy a Δ_2-condition near infinity if there exists $t_0 > 0$ such that (1.18.6) holds for every $t \geq t_0$. Let Ω be a domain of $\mathbb{R}^N$ and let A be an $N-$function. The Orlicz class K_A is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions u defined on Ω and satisfying

$$\int_\Omega A(|u(x)|)dx < \infty.$$

Since A is convex $K_A(\Omega)$ is always a convex set of functions but is may not be a vector space, $K_A(\Omega)$ is a vector space (under pointwise addition and scalar multiplication) if and only if (A, Ω) is $\Delta-$regular (i.e. A satisfies a global Δ_2-condition or Δ_2-condition near infinity and Ω has finite volume). The Orlicz space L_A is defined to be the linear hull of the Orlicz class $K_A(\Omega)$, that is, the smallest vector space containing $K_A(\Omega)$. Evidently L_A consists of all scalar multiples λu of elements $u \in K_A(\Omega)$. Thus $K_A(\Omega) \subset L_A(\Omega)$, these sets being equal if and only if (A, Ω) is $\Delta-$regular. The reader may verify that the functional

$$\|u\|_A = \|u\|_{A,\Omega} = \inf\left\{k > 0, \ \int_\Omega A(\left|\frac{u(x)}{k}\right|)dx \leq 1\right\} \tag{1.18.7}$$

is a norm on $L_A(\Omega)$ (this norm is due to Luxembourg [**Lux**]). For $\|u\|_A > 0$ the infimum in (1.18.7) is attained in $k = \|u\|_A$. $L_A(\Omega)$ is a Banach space with respect to the norm (1.18.7). If A and $\tilde{A}$ are complementary $N-$functions, a generalized version of Hölder inequality

$$\left|\int_\Omega u(x)v(x)dx\right| \leq 2\|u\|\|_{A,\Omega}\|v\|_{\tilde{A},\Omega}. \tag{1.18.8}$$

A sequence (u_j) of functions in $L_A(\Omega)$ is said to converge in mean to $u \in L_A(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega A(|u_j(x) - u(x)|)dx = 0.$$

Let $E_A(\Omega)$ denote the closure in $L_A(\Omega)$ of the space of functions u which are bounded on Ω and have bounded support in $\overline{\Omega}$. Therefore, if (A, Ω) is $\Delta-$regular, then $E_A(\Omega) = K_A(\Omega) = L_A(\Omega)$. If (A, Ω) is not $\Delta-$regular, we have

$$E_A(\Omega) \subset K_A(\Omega) \subseteq L_A(\Omega), \tag{1.18.9}$$

so that $E_A(\Omega)$ is a proper closed subspace of $L_A(\Omega)$ and a maximal linear subspace of $K_A(\Omega)$ in this case. For fixed $v \in L_{\tilde{A}(\Omega)}$ the linear functional L_v defined by $L_v(u) = \int_\Omega u(x)v(x)dx$ belongs to $(L_A(\Omega))'$. Denoting by

$\|L_v\|$ its norm in that space, we have

$$\|v\|_{\tilde{A}} \leq \|L_v\| \leq 2\|v\|_{\tilde{A}}. \tag{1.18.10}$$

Then, the dual space $(E_A(\Omega))'$ of $E_A(\Omega)$ is isomorphic and homeomorphic to $L_{\tilde{A}}(\Omega)$ and $L_A(\Omega)$ is reflexive if and only if both (A,Ω) and $(\tilde{A},\Omega)$ are $\Delta-$regular.

For a given domain Ω in $\mathbb{R}^N$ and a given defining $N-$function A, the Orlicz-Sobolev space $W^m L_A(\Omega)$ consists of those functions u in $L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ also belong to $L_A(\Omega)$ for all α with $|\alpha| \leq m$. The space $W^m E_A(\Omega)$ is defined in analogous fashion, $W^m L_A(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,A} = \|u\|_{m,A,\Omega} = \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{A,\Omega}, \tag{1.18.11}$$

and that $W^m E_A(\Omega)$ is closed subspace of $W^m L_A(\Omega)$, and coincides if and only if (A,Ω) is $\Delta-$regular. As in the case of Lebesgue-Sobolev spaces, $W_0^m L_A(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^m L_A(\Omega)$ with analogous definition for $W_0^m E_A(\Omega)$. Many properties of Orlicz-Sobolev spaces are obtained by very straightforward generalization of the proofs of the same properties for Lebesgue-Sobolev spaces. We summarize some of these in the following theorem

Theorem 1.66. *We have*
(a) $W^m E_A(\Omega)$ *is separable.*
(b) If (A,Ω) *and* $(\tilde{A},\Omega)$ *are* $\Delta-$*regular, then* $W^m E_A(\Omega) = W^m L_A(\Omega)$ *is reflexive.*
(c) Each element L *of the dual space* $(W^m E_A(\Omega))'$ *is given by*

$$L(u) = \sum_{0\leq|\alpha|\leq m} \int_\Omega D^\alpha u(x) v_\alpha(x) dx \text{ for some functions } v_\alpha \in L_{\tilde{A}}(\Omega),\ 0 \leq |\alpha \leq m.$$

(d) $C^\infty(\Omega) \cap W^m E_A(\Omega)$ *is dense in* $W^m E_A(\Omega)$.
(e) If Ω *has the segment property, then* $C^\infty(\overline{\Omega})$ *is dense in* $W^m E_A(\Omega)$.
(f) $C_0^\infty(\mathbb{R}^N)$ *is dense in* $W^m E_A(\mathbb{R}^N)$. *Thus* $W_0^m L_A(\mathbb{R}^N) = W^m E_A(\mathbb{R}^N)$.

Proof. See **[A]**, Theorem 8.28. □

Now, let A be a given $N-$function, we shall always suppose that

$$\int_0^1 \frac{A^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty, \tag{1.18.12}$$

replacing, if necessary, A by another $N-$function equivalent to A near infinity. Suppose also that

$$\int_1^\infty \frac{A^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty. \tag{1.18.13}$$

For instance, if $A = \frac{t^p}{p}$, then (1.18.13) holds precisely when $p \leq N$. With (1.18.13) satisfied we define the *Sobolev conjugate* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau, \quad t \geq 0. \tag{1.18.14}$$

It may readily be checked that A_* is an $N-$function. If $1 < p < N$, we have, setting $q = \frac{Np}{N-p}$,

$$A_{p^*}(t) = q^{1-q} p^{-\frac{q}{p}} A_q(t).$$

It is also readily seen for the case $p = N$ that $A_{N*}(t)$ is equivalent near infinity to the $N-$function $e^t - t - 1$.

CHAPTER 2

Quasilinear elliptic problems with general measure data and variable exponent

Recently, Sanchón and Urbano [**SU**] studied a Dirichlet problem of the $p(x)-$Laplacian equation and obtained the existence and uniqueness of entropy solutions for L^1-data, as well as integrability results for the solution and its gradient. The proofs rely crucially on a priori estimates in Marcinkiewicz spaces with variable exponents. Besides, Bendahmane and Wittbold in [**BW**] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data, and Zimmermann with Wittbold have already studied the corresponding elliptic problem for more general elliptic equations involving lower order terms in [**Zha**], taking into account a measure μ in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. As far as we know, there are few papers concerned with right-hand side measure [**ABR, YAR**] and references therein. Recalling that the notion of renormalized solution was used in [**Al**], to get the existence of solution in Orlicz-Sobolev spaces under the assumption that μ is general. This Chapter is organized as follows. In Section 2.1, we recall the definition of $p(\cdot)-$capacity and establish their relations with measures. In Section 2.2, some basic assumptions and properties of measures are recalled and a new formulation of renormalized solutions with the main result are proposed. In Section 2.3, we obtain a priori estimates for renormalized solutions and its weak gradients using approximate problems with regular data. Finally, in Section 2.4, we consider cut-off test functions and, using the a priori estimates, we establish the existence result.

2.1. Elliptic $p(\cdot)-$capacity and general measures

The notion of $p(\cdot)-$capacity plays the expected role in the potential theory and in the study of Sobolev functions in the variable exponent setting, see [**HHK, HHKV1, HHKV2, HL**]. In general, the $p(\cdot)-$capacity is used to measure finite properties of functions and sets. Then $p(\cdot)-$capacity enjoys the usual fine properties of capacity when $1 < p_- \leq p_+ < \infty$, see [**HHKV1, DHHR**], some of the properties remain still open for the case $p_- = 1$. In this part, we study Lebesgue points and quasi-continuity of Sobolev functions in the variable exponent setting. In [**HH**] (these are extensions of the classical results in [**HKM**]), the authors proved that every Sobolev function has Lebesgue points outside of a set of $p(\cdot)-$capacity zero and that the precise pointwise representative of a Sobolev function is $p(\cdot)$ quasi-continuous. First we introduce the basic tools that we need in our study

Definition 2.1. Let $p(\cdot) : \Omega \to [1, \infty)$ be variable exponent. The $p(\cdot)-$capacity of a set $E \subset \mathbb{R}^N$ is defined as

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^N} |u|^{p(x)} + |\nabla u|^{p(x)} dx,$$

where the infimum is taken over admissible functions $u \in S_{p(\cdot)}(E)$ where

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^N) : u \geq 1 \text{ in an open set containing } E\}.$$

It is easy to see that if we restrict these admissible functions $S_{p(\cdot)}(E)$ to the case $0 \leq u \leq 1$, we get the same capacity.

Definition 2.2. We say that a claim holds $p(\cdot)$ quasi-everywhere if it holds everywhere except in a set of $p(\cdot)-$capacity zero. A function $u : \Omega \to \mathbb{R}$ is said to be $p(\cdot)$ quasi-continuous if for every $\epsilon > 0$ there exists an open U with $C_{p(\cdot)}(U) < \epsilon$ such that u restricted to $\Omega \backslash U$ is continuous.

A variable exponent version of the relative $p(\cdot)-$capacity of the condenser has been used in **[HHK]**. This alternative capacity of a set is taken relative to a surrounding open subset of $\mathbb{R}^N$. Suppose that $p_+ < \infty$ and $p(x)$ satisfies the Log-Hölder continuity condition (1.17.3) and let K be a compact subset of Ω. The relative $p(\cdot)-$capacity of K in Ω is the number

$$\mathrm{cap}_{p(\cdot)}(K,\Omega) = \inf \left\{ \int_\Omega |\nabla \varphi|^{p(x)} dx : \varphi \in C_0^\infty(\Omega) \text{ and } \varphi \geq 1 \text{ in } K \right\}.$$

For an open set $U \subset \Omega$ we define

$$\mathrm{cap}_{p(\cdot)}(U,\Omega) = \sup \{\mathrm{cap}_{p(\cdot)}(K,\Omega) : K \subset U \text{ compact}\}$$

and for an arbitrary $E \subset \Omega$

$$\mathrm{cap}_{p(\cdot)}(E,\Omega) = \inf \{\mathrm{cap}_{p(\cdot)}(U,\Omega) : U \supset E \text{ open}\}$$

Then

$$\mathrm{cap}_{p(\cdot)}(E,\Omega) = \sup \{\mathrm{cap}_{p(\cdot)}(K,\Omega) : K \supset E \text{ compact}\}$$

for all Borel sets $E \subset \Omega$.

Definition 2.3. We say that $u : \Omega \to \overline{\mathbb{R}}$ is $p(\cdot)$ quasi-continuous if for $\epsilon > 0$ there exists an open set $A \subset \Omega$ with $\mathrm{cap}_{p(\cdot)}(A,\Omega) \leq \epsilon$, such that $u_{(\Omega \setminus A)}$ is continuous. Every $u \in W^{1,p(\cdot)}(\Omega)$ has a $p(\cdot)$ quasi-continuous representative, always denoted in this paper by u, which is essentially unique.

Denote by $\mathcal{M}_b(\Omega)$ the space of all signed measures on Ω, i.e., the space of all $\sigma-$additive set functions μ with values in $\mathbb{R}$ defined on the Borel $\sigma-$algebra. If μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|$ (the total variation of μ) is a bounded positive measure on Ω. The positive part, the negative part, and the total variation of a measure μ in $\mathcal{M}_b(\Omega)$ are denoted by μ^+, μ^-, and $|\mu|$, respectively. We recall that for a measure μ in $\mathcal{M}_b(\Omega)$, and a Borel set $E \subseteq \Omega$, the restriction of μ in E is the measure $\mu \perp E$ defined by $(\mu \perp E)(B) = \mu(E \cap B)$ for any Borel set $B \subseteq \Omega$. We will denote by $\mathcal{M}_0(\Omega)$ the space of all measures μ in $\mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for every set E satisfying $\mathrm{cap}_{p(\cdot)}(E,\Omega) = 0$. Examples of measures in $\mathcal{M}_0(\Omega)$ are the $L^1(\Omega)-$functions, or the measures in $W^{-1,p'(\cdot)}(\Omega)$. Next we have a decomposition of a measure in $\mathcal{M}_0(\Omega)$.

Proposition 2.4. *Let $\mu \in \mathcal{M}_b(\Omega)$ and assume that $p(x)$ satisfies Log-Hölder condition (1.17.3) with $1 < p_- \leq p_+ < \infty$. Then $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ if and only if $\mu \in \mathcal{M}_0(\Omega)$. Thus, if $\mu \in \mathcal{M}_0(\Omega)$, there exist $f \in L^1(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, such that*

$$\mu = f - \mathrm{div}(g),$$

in the sense of distributions.

Proof. See **[Zha]**, Proposition 2.6. □

We denote by $\mathcal{M}_s(\Omega)$ the set of all measures $\mu \in \mathcal{M}_b(\Omega)$ such that there exists a Borel set $E \subset \Omega$, with $\mathrm{cap}_{p(\cdot)}(E,\Omega) = 0$, and such that $\mu = \mu \perp E$. The measures μ_0 and μ_s will be called the absolutely continuous and the singular parts of μ with respect to the $p(\cdot)-$capacity. So, if $\mu \in \mathcal{M}_b(\Omega)$, thanks to decomposition result (i.e., Proposition 2.4), we can split it into a sum (uniquely determined) of its absolutely continuous part μ_0 with respect to $p(\cdot)-$capacity, and its singular part μ_s, that is μ_s is concentrated on a set E of zero $p(\cdot)-$capacity. Hence, if $\mu \in \mathcal{M}_b(\Omega)$, we have

$$\mu = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^-, \tag{2.1.1}$$

in the sense of distributions, for some $f \in L^1(\Omega)$, $g \in (L^{p'(\cdot)}(\Omega))^N$, where μ_s^+ and μ_s^- are respectively the positive and the negative part of μ_s, note that the decomposition of the singular part of μ are nonnegative measures which are concentrated on two disjoint subsets E^+ and E^- with $E = E^+ \cup E^-$.

2.2. General assumptions, renormalized formulation and main result

As we said before, the main purpose of this paper is to extend the results in **[DMOP]** to a non-constant $p(\cdot)$. Defining the truncation function T_k by

$$T_k(s) := \max\{-k, \min\{k, s\}\}, \quad s \in \mathbb{R},$$

let us consider the space $\mathcal{T}_0^{1,p(\cdot)}(\Omega)$ of all functions $u : \Omega \to \overline{\mathbb{R}}$ which are measurable and finite a.e. in Ω, and such that $T_k(u)$ belongs to $W_0^{1,p(\cdot)}(\Omega)$ for every $k > 0$. It is easy to see that every function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ has a $\mathrm{cap}_{p(\cdot)}$ quasi-continuous representative, that will always be identified with u. Moreover, for every $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$, which is unique up to almost everywhere equivalence, such that $\nabla T_k(u) = v\chi_{\{|u|\le k\}}$ a.e. in Ω, for every $k > 0$, (see **[B6]**, Lemma 2.1). Hence it is possible to define a generalized gradient ∇u of u, setting $\nabla u = v$. If $u \in L^1_{loc}(\Omega)$, this gradient may differ from the distributional gradient of u, while it coincides with the usual gradient for every $u \in W^{1,1}(\Omega)$. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and consider the elliptic problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.1}$$

where μ is a Radon measure with bounded variation on Ω and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, $a(\cdot, \zeta)$ is measurable in Ω, for every $\zeta \in \mathbb{R}^N$, and $a(x, \cdot)$ is continuous in $\mathbb{R}^N$, for almost every $x \in \Omega$), such that the following assumptions hold

$$a(x, \zeta) \cdot \zeta \ge b|\zeta|^{p(x)}, \tag{2.2.2}$$

for almost every $x \in \Omega$ and for every $\zeta \in \mathbb{R}^N$, where b is a positive constant;

$$|a(x, \zeta)| \le \beta(j(x) + |\zeta|^{p(x)-1}), \tag{2.2.3}$$

for almost every $x \in \Omega$ and for every $\zeta \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ and $\beta > 0$;

$$[a(x, \zeta) - a(x, \zeta')] \cdot [\zeta - \zeta'] > 0, \tag{2.2.4}$$

for almost every $x \in \Omega$ and for every $\zeta, \zeta' \in \mathbb{R}^N$, with $\zeta \ne \zeta'$.

Hypotheses $(2.2.2) - (2.2.4)$ are the natural extensions of the classical assumptions in the study of nonlinear monotone operators in divergence form for constant $p(\cdot) \equiv p$ **[LL, KR]**. These assumptions allow us, in particular, to exploit the functional analytical properties of Lebesgue and Sobolev spaces with variable exponent (see Section 1.17) arising in the study of problem (2.2.1). A weak solution of (2.2.1) is a function $u \in W_0^{1,1}(\Omega)$ such that $a(x, \nabla u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega a(x, \nabla u) \cdot \nabla\varphi dx = \int_\Omega \varphi d\mu, \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2.2.5}$$

A weak energy solution is a weak solution such that $u \in W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.5. If $\mu \in W^{-1,p'(\cdot)}(\Omega)$, it is well known that problem (2.2.1) has a unique solution $u \in W_0^{1,p(\cdot)}(\Omega)$, in the variational sense, that is

$$\int_\Omega a(x, \nabla u) \cdot \nabla\varphi \, dx = \langle \mu, \varphi \rangle_{W^{-1,p'(\cdot)}(\Omega), W_0^{1,p(\cdot)}(\Omega)},$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, notice that if $p(\cdot) > N$, then $\mathcal{M}_b(\Omega)$ is a subset of $W^{-1,p'(\cdot)}(\Omega)$.

The model case for (2.2.1) is the Dirichlet problem for the $p(x)-$Laplacian operator $\Delta_{p(x)}(u) = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$

$$\begin{cases} -\Delta_{p(x)} u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.6}$$

We start by extending the notion of renormalized solution to problem (2.2.1) as follows

Definition 2.6. A measurable function u is a renormalized solution to problem (2.2.1) if the following conditions hold :

(a) $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$,

(b) $|\nabla u|^{p(\cdot)-1}$ belongs to $L^{q(\cdot)}(\Omega)$, for every $q(\cdot) < \frac{N}{N-1}$,

(c) if w belongs to $W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and if there exist $k > 0$, and $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r(\cdot)}(\Omega) \cap L^\infty(\Omega)$, with $r(\cdot) > N$, such that

$$\begin{aligned} w &= w^{+\infty} \quad \text{a.e on the set } \{u > k\}, \\ w &= w^{-\infty} \quad \text{a.e on the set } \{u < k\}. \end{aligned} \tag{2.2.7}$$

Then

$$\int_\Omega a(x, \nabla u) \cdot \nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega w^{+\infty} d\mu_s^+ - \int_\Omega w^{-\infty} d\mu_s^-. \tag{2.2.8}$$

DEFINITION 2.7. Let $W^{1,\infty}(\mathbb{R})$ be the set of all bounded Lipschitz continuous functions $h : \mathbb{R} \to \mathbb{R}$ whose derivative h' has compact support. Clearly every function $h \in W^{1,\infty}(\mathbb{R})$ is constant outside the support of its derivatives, so that we can define the constants

$$h(+\infty) = \lim_{t\to+\infty} h(t), \quad h(-\infty) = \lim_{t\to-\infty} h(t).$$

REMARK 2.8. Notice that, if u is a renormalized solution of (2.2.1), then for every $h \in W^{1,\infty}(\mathbb{R})$ and for every $\varphi \in W^{1,r(\cdot)}(\Omega) \cap L^\infty(\Omega)$, with $r(\cdot) > N$, such that $h(u)\varphi$ belongs to $W_0^{1,p(\cdot)}(\Omega)$, the function $w = h(u)\varphi$ satisfies all the requirements in (c). Hence we can put it as test function in (2.2.8), obtaining

$$\begin{aligned} &\int_\Omega a(x, \nabla u) \cdot \nabla u h'(u) \varphi dx + \int_\Omega a(x, \nabla u) \cdot \nabla \varphi h(u) dx \\ &= \int_\Omega h(u) \varphi d\mu_0 + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-. \end{aligned} \tag{2.2.9}$$

If h has compact support, (2.2.9) becomes

$$\int_\Omega a(x, \nabla u) \cdot \nabla u h'(u) \varphi dx + \int_\Omega a(x, \nabla u) \cdot \nabla \varphi h(u) dx = \int_\Omega h(u) \varphi d\mu_0, \tag{2.2.10}$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence, since $\mu_0 \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ and by a density argument, (2.2.10) holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Our main result is

THEOREM 2.9. *Assume* (2.2.2) $-$ (2.2.4) *and* $\mu \in \mathcal{M}_b(\Omega)$. *There exists a renormalized solution* u *to the problem* (2.2.1).

2.3. A priori estimates and compactnes results

To prove the main result we have to obtain the a priori estimates for renormalized solutions in Lebesgue-Sobolev spaces with variable exponent. From these estimates we derive uniform bounds for solution and its weak gradients (see Lemmas 2.10, 2.11). Finally, the existence is obtained by passing to the limit in a sequences of weak energy solutions of adequate approximated problems. Let us first show the following interesting property of renormalized solutions; throughout the paper C will indicate any positive constant whose value may change from line to line.

LEMMA 2.10. *Let* $p(x) \in C_+(\overline{\Omega})$ *with* $1 < p_- \le p_+ < N$ *satisfy the Log-Hölder continuity condition* (1.17.3) *and let* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *be such that*

$$\frac{1}{k} \int_\Omega |\nabla T_k(u)|^{p(x)} dx \le M, \tag{2.3.1}$$

for every $k > 0$. *Then there exist* $C = C(N, M, p_-) > 0$ *such that*

$$\operatorname{meas}(\{|u| > k\}) \le C k^{-\frac{N(p_- - 1)}{N-1}}. \tag{2.3.2}$$

PROOF. Recalling the Sobolev embedding Theorem in Proposition 1.62, we have the continuous embedding
$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{(p^*)_-}(\Omega),$$
where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $(p^*)_- = \frac{Np_-}{N-p_-}$. It follows from the last continuous embedding that, for every $k > 1$, nothing that $\{|u| \geq k\} = \{|T_k(u)| \geq k\}$. Hence
$$\|T_k(u)\|_{(p^*)_-} \leq C\|\nabla T_k(u)\|_{p(x)} \leq C\big(\int_\Omega |\nabla T_k(u)|^{p(x)}dx\big)^\alpha \leq C\big(Mk\big)^\alpha$$
where
$$\alpha = \begin{cases} \frac{1}{p_-} & \text{if } |\nabla T_k(u)|^{p(x)} \geq 1, \\ \frac{1}{p_+} & \text{if } |\nabla T_k(u)|^{p(x)} \leq 1. \end{cases}$$
Then
$$\text{meas}\{|u| > k\} \leq \left(\frac{\|T_k(u)\|_{(p^*)_-}}{k}\right)^{(p^*)_-} \leq CM^{\frac{(p^*)_-}{p_-}} k^{\frac{(p^*)_-}{p_-} - (p^*)_-} \leq Ck^{\frac{-N(p_- - 1)}{N-p_-}},$$
this proves that u satisfies (2.3.2). □

LEMMA 2.11. *Let $p(x) \in C_+(\overline{\Omega})$ with $1 < p_- \leq p_+ < N$ satisfy the Log-Hölder continuity condition* (1.17.3) *and assume that $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ satisfies* (2.3.1) *for every k. Then for every $h > 0$*

$$\text{meas}(\{|\nabla u| > k\}) \leq C(N,p_-)M^{\frac{N}{N-1}} h^{-\frac{N(p_- - 1)}{N-p_-}}. \tag{2.3.3}$$

PROOF. For $k, \lambda \geq 0$, set
$$\Phi(k,\lambda) = \text{meas}\{|\nabla u|^{p_-} > \lambda,\ |u| > k\}.$$
According to Lemma 2.10, we have
$$\Phi(k,0) \leq C(N,p_-)M^{\frac{N}{N-p_-}} k^{\frac{-N(p_- - 1)}{N-p_-}}, \text{ for all } k \geq 1. \tag{2.3.4}$$
Using the fact that the function $\lambda \mapsto \Phi(k,\lambda)$ is non-increasing, we get for $k > 0$ and $\lambda > 0$ that
$$\begin{aligned} \Phi(0,\lambda) &= \frac{1}{\lambda}\int_0^\lambda \Phi(0,\lambda)ds \leq \frac{1}{\lambda}\int_0^\lambda \Phi(0,s)ds \\ &\leq \frac{1}{\lambda}\int_0^\lambda [\Phi(0,s) + (\Phi(k,0) - \Phi(k,s)]dx \\ &\leq \Phi(k,0) + \frac{1}{\lambda}\int_0^\lambda (\Phi(0,s) - \Phi(k,s))ds. \end{aligned} \tag{2.3.5}$$
Observe that since
$$\Phi(0,s) - \Phi(k,s) = \text{meas}\{|u| \leq k,\ |\nabla u|^{p_-} > s\}$$
and using (2.3.1), we obtain
$$\int_0^\infty (\Phi(0,s) - \Phi(k,s))ds = \int_{\{|u|<k\}} |\nabla u|^{p_-}dx \leq Mk. \tag{2.3.6}$$
Going back to (2.3.5) and using (2.3.4) and (2.3.6) we arrive at
$$\Phi(0,\lambda) \leq \frac{Mk}{\lambda} + C(N,p_-)M^{\frac{N}{N-p_-}} k^{\frac{-N(p_- - 1)}{N-p_-}}, \tag{2.3.7}$$
for all $k \geq 1, \lambda > 0$. The minimization of (2.3.7) in k and setting $\lambda = h^{p_-}$ gives (2.3.3). □

For the critical case $p_+ = N$, the problem is well posed in the energy class $W_0^{1,N}(\Omega)$ for the second member $\mu \in W^{-1,N'}$. On the other hand, for $f \in L^1(\Omega)$ the theory of **[DHHR]** can be adapted.

REMARK 2.12. We remark that, as a consequence of estimates (2.3.2) and (2.3.3), we can improve the following results if $p_+ = N$
$$\text{meas}(\{|v| > k\}) \leq Ck^{-r_-(p_- - 1)}, \text{ for all } k > 0,\ r(\cdot) > 1,$$
$$\text{meas}(\{|\nabla v| > k\}) \leq Ck^{-s_-} \text{ for every } s(\cdot) < N.$$

Now, let us come back to the existence of a renormalized solution for problem (2.2.1), as we said before, if $\mu \in \mathcal{M}_b(\Omega)$ we can split it in this way

$$\mu = \mu_0 + \mu_s = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

for some $f \in L^1(\Omega)$, $g \in (L^{p'(\cdot)}(\Omega))^N$, and $\mu_s \in M_s(\Omega)$, that is, μ_s is concentrated on a set $E \subset \Omega$ with $\mathrm{cap}_{p(\cdot)}(E) = 0$ and such that $\mu = \mu \perp E$. There are many ways to approximate this measure looking for existence of solutions for problem (2.2.1), we will make the following choice

$$\mu_\epsilon = f_\epsilon - \operatorname{div}(g_\epsilon) + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus, \tag{2.3.8}$$

where

$$\begin{gathered} f_\epsilon \text{ is a sequence of functions in } L^1(\Omega) \text{ such that} \\ f_\epsilon \to f \text{ in } L^1(\Omega) \text{ weakly,} \end{gathered} \tag{2.3.9}$$

$$\begin{gathered} g_\epsilon \text{ is a sequence of functions in } (L^{p'(\cdot)}(\Omega))^N \text{ such that} \\ g_\epsilon \to g \text{ in } (L^{p'(\cdot)}(\Omega))^N \text{ strongly,} \end{gathered} \tag{2.3.10}$$

$$\begin{gathered} \lambda_\epsilon^\oplus \text{ is a a non-negative measure in } \mathcal{M}_b(\Omega) \text{ such that} \\ \lambda_\epsilon^\oplus \to \mu_s^+ \text{ in the narrow topology,} \end{gathered} \tag{2.3.11}$$

$$\begin{gathered} \lambda_\epsilon^\ominus \text{ is a a non-negative measure in } \mathcal{M}_b(\Omega) \text{ such that} \\ \lambda_\epsilon^\ominus \to \mu_s^- \text{ in the narrow topology.} \end{gathered} \tag{2.3.12}$$

Notice that this approximation can be easily obtained via a standard convolution argument. Then, it is easy to see that, if (μ_ϵ) has a splitting converging to μ, then (μ_ϵ) converges weakly–* in $\mathcal{M}_b(\Omega)$ to μ, so that there exists $M > 0$ such that

$$\|\mu_\epsilon\|_{L^1(\Omega)} \le C|\mu| \le M \quad \forall \epsilon > 0.$$

Observe that we can decompose $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ in the following

$$\lambda_\epsilon^\oplus = \lambda_{\epsilon,0}^\oplus + \lambda_{\epsilon,s}^\oplus, \quad \lambda_\epsilon^\ominus = \lambda_{\epsilon,0}^\ominus + \lambda_{\epsilon,s}^\ominus$$

with

$$\begin{aligned} &\lambda_{\epsilon,0}^\oplus, \lambda_{\epsilon,0}^\ominus \in \mathcal{M}_0(\Omega), \quad \lambda_{\epsilon,0}^\oplus, \lambda_{\epsilon,0}^\ominus \ge 0, \\ &\lambda_{\epsilon,s}^\oplus, \lambda_{\epsilon,s}^\ominus \in \mathcal{M}_s(\Omega), \quad \lambda_{\epsilon,s}^\oplus, \lambda_{\epsilon,s}^\ominus \ge 0. \end{aligned}$$

On the other hand, the measure μ_ϵ can be decomposed as

$$\mu_\epsilon = \mu_{\epsilon,0} + \mu_{\epsilon,s} = \mu_{\epsilon,0} + \mu_{\epsilon,s}^+ - \mu_{\epsilon,s}^-,$$

where $\mu_{\epsilon,0}$ is a measure in $\mathcal{M}_0(\Omega)$ and where $\mu_{\epsilon,s}^+$ and $\mu_{\epsilon,s}^-$ (the positive and the negative parts of $\mu_{\epsilon,s}$) are two nonnegative measures in $\mathcal{M}_s(\Omega)$, which are concentrated on two disjoint subsets E_s^+ and E_s^- of zero $p(\cdot)-$capacity. Therefore we can conclude, by the definition of μ_ϵ, that

$$\mu_{\epsilon,0} = f_\epsilon - \operatorname{div}(g_\epsilon) + \lambda_{\epsilon,0}^\oplus - \lambda_{\epsilon,0}^\ominus, \quad \mu_{\epsilon,s} = \lambda_{\epsilon,s}^\oplus - \lambda_{\epsilon,s}^\ominus. \tag{2.3.13}$$

In particular, we have

$$0 \le \mu_{\epsilon,s}^+ \le \lambda_{\epsilon,s}^\oplus, \quad 0 \le \mu_{\epsilon,s}^- \le \lambda_{\epsilon,s}^\ominus.$$

Let us call u_ϵ the solution of problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u_\epsilon)) = \mu_\epsilon & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.14}$$

that exists and is unique **[LL]**, and let recall that u_ϵ is a renormalized solution of (2.3.14) with μ_ϵ as data. Hence it satisfies

$$\begin{aligned} \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla w \, dx = \int_\Omega w f_\epsilon dx + \langle -\operatorname{div}(g_\epsilon), w \rangle + \int_\Omega w \, d(\lambda_{\epsilon,0}^\oplus - \lambda_{\epsilon,0}^\ominus) \\ + \int_\Omega w^{+\infty} d\mu_{\epsilon,s}^+ - \int_\Omega w^{-\infty} d\mu_{\epsilon,s}^-, \end{aligned} \tag{2.3.15}$$

for all $w \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and such that there exist $k > 0$, $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r(\cdot)}(\Omega) \cap L^\infty(\Omega)$, with $r(\cdot) > N$, such that $w = w^{+\infty}$ a.e. on the set $\{u_\epsilon > k\}$ and $w = w^{-\infty}$ a.e. on the set $\{u_\epsilon < -k\}$. Note that L^1-compactness results for the gradients of a sequence of approximate solutions of nonlinear equations have been obtained in [**BG1**], and we emphasize that the first result is contained in a pioneering work by Leray-Lions [**LL**]. As a first step, we find a function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ which is the limit, up to a subsequence, of (u_ϵ) in suitable topologies.

PROPOSITION 2.13. *Let (u_ϵ) be a sequence of renormalized solutions of* (2.3.14). *Then there exists $M > 0$ such that*

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)} dx \le Mk, \tag{2.3.16}$$

for every ϵ and every $k > 0$. Moreover, there exists a measurable function u such that $T_k(u)$ belongs to $L^{p(\cdot)}(\Omega)$, and, up to a subsequence, for any $k > 0$, we have

(i) *$u_\epsilon \to u$ a.e. on Ω and strongly in $L^1(\Omega)$,*
(ii) *$T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega)$ and strongly in $L^1(\Omega)$,*
(iii) *$\nabla u_\epsilon \to \nabla u$ a.e. on Ω.*
(iv) *$a(x, \nabla u_\epsilon) \to a(x, \nabla u)$ in $(L^{q(\cdot)}(\Omega))^N$ for every $1 \le q(\cdot) < \frac{N}{N-1}$.*

PROOF. Let us choose $w = T_k(u_\epsilon)$ as a admissible test function in (2.3.15). We obtain

$$\int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla T_k(u_\epsilon) dx = \int_\Omega T_k(u_\epsilon) d\mu_\epsilon = \int_\Omega T_k(u_\epsilon) d\mu_\epsilon.$$

Since $|T_k(u_\epsilon)| \le k$, the previous identity implies by (2.2.2)

$$C_0 \int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)} dx \le k|\mu_\epsilon(\Omega)| \le Mk, \tag{2.3.17}$$

where the constants C_0 and M do not depend on ϵ.

(i) We prove now that the sequence (u_ϵ) admits a subsequence which converges to a function u. Using (2.3.17) we see that $(\nabla T_k(u_\epsilon))_\epsilon$ is bounded in $L^{p(\cdot)}(\Omega)$ for every $k > 0$, we also have by (2.3.1), that $\text{meas}\{|u|_\epsilon > k\}$ is finite for every $k > 0$. Let us prove that, up to a subsequence, (u_ϵ) is a Cauchy sequence in measure (i.e. $u_\epsilon \to u$ in measure) in Ω. We have

$$\{|u_\epsilon - u_{\epsilon'}| > t\} \subseteq \{|u_\epsilon| > k\} \cup \{|u_{\epsilon'}| > k\} \cup \{|T_k(u_\epsilon) - T_k(u_{\epsilon'})| > t\}$$

for every $\epsilon, \epsilon' \in \mathbb{N}$. So that

$$\begin{aligned} \text{meas}\{|u_\epsilon - u_{\epsilon'}| > t\} &\le \text{meas}\{|u_\epsilon| > k\} \\ &+ \text{meas}\{|u_{\epsilon'}| > k\} + \text{meas}\{|T_k(u_\epsilon) - T_k(u_{\epsilon'})| > t\}. \end{aligned}$$

for every fixed $\delta > 0$, by the first estimate (2.3.2) there exists $k_0(\delta) > 0$ such that

$$\text{meas}\{|u_\epsilon| > k\} + \text{meas}\{|u_{\epsilon'}| > k\} < \frac{\delta}{2},$$

for every $k > k_0$ and for every $\epsilon, \epsilon' \in \mathbb{N}$. Let now $k > k_0$ be fixed. Thanks to (2.3.17), the sequence $(T_k(u_\epsilon))$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, and then we can extract a subsequence of $(T_k(u_\epsilon))$ (depending of k) converging strongly in $L^{q(\cdot)}(\Omega)$ for every $1 \le q(\cdot) \ll p^*(\cdot)$ (i.e., $(p^* - q)_- > 0$), we obtain a subsequence, still denoted by $(T_k(u_\epsilon))$, converging strongly in $L^{q(\cdot)}(\Omega)$ for every $1 \le q(\cdot) \ll p^*(\cdot)$ and which turns out to be a Cauchy sequence in measure. Then there exist $n_0 \in \mathbb{N}$ such that

$$\text{meas}(\{|T_k(u_\epsilon) - T_k(u_{\epsilon'})| > t)\} \le \int_\Omega \left(|\frac{T_k(u_\epsilon) - T_k(u_{\epsilon'})}{t}| \right)^{q(x)} dx \le \frac{\delta}{2},$$

for every $\epsilon, \epsilon' > n_0(k,t)$. Collecting latest informations we obtain that, up to a subsequence, (u_ϵ) is a Cauchy sequence in measure, hence that $u_\epsilon \to u$ in measure.

(ii) In the previous step we have obtained that $T_k(u_\epsilon)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, for every fixed k, then we can extract a subsequence (still denoted by $T_k(u_\epsilon)$) converging to a function ν_k weakly in $W_0^{1,p(\cdot)}(\Omega)$. Since

$T_k(s)$ is continuous and u_ϵ converges to u almost everywhere in Ω (by (i)). Then $\nu_k = T_k(u)$, in conclusion $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ and

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \quad \forall k > 0,$$

in addition we have $\int_\Omega |\nabla T_k(u)|^p dx \leq Mk$, where M is the constant defined in (2.3.1) and u satisfies the estimates (2.3.2) and (2.3.3).

(iii) Before proving ∇u_ϵ is a Cauchy sequence in measure we recall that μ_0-compactness results for the gradients are similar to the one obtained in $L^1(\Omega)$, and we emphasize that this result was generalized in Sobolev spaces with variable exponent in **[BW]**. In the proof we will need the following standard result

LEMMA 2.14. **[Hal]** *Let $(X, \mathcal{M}, m)$ a measurable space, such that $m(X) < +\infty$. Let γ be a measurable function, $\gamma : X \to [0, +\infty)$ such that $m(\{x \in X, \gamma(x) = 0\}) = 0$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$m(A) \leq \epsilon, \ \forall A \in \mathcal{M} \text{ with } \int_A \gamma \, dm \leq \delta.$$

Our proof relies on the following claim

$$\nabla u_\epsilon \to \nabla u \text{ in measure.} \tag{2.3.18}$$

In order to prove (2.3.18), given $t > 0$, for every η and $k > 0$ ($\epsilon, \epsilon' \in \mathbb{N}$),

$$E_1 = \{x \in \Omega : |\nabla u_\epsilon(x)| > k\} \cup \{x \in \Omega : |\nabla u_{\epsilon'}(x)| > k\},$$
$$E_2 = \{|u_\epsilon - u_{\epsilon'}| > \eta\},$$
$$E_3 = \{x \in \Omega : |u_\epsilon(x) - u_{\epsilon'}(x)| \leq \eta, \ |\nabla u_\epsilon(x)| \leq k, \ |\nabla u_{\epsilon'}| \leq k, \ |\nabla u_\epsilon - \nabla u_{\epsilon'}| \geq t\}.$$

Remark that

$$\{x \in \Omega : |\nabla u_\epsilon - \nabla u_{\epsilon'})(x)| \geq t\} \subset E_1 \cup E_2 \cup E_3. \tag{2.3.19}$$

Since (u_ϵ) and $(\nabla u_{\epsilon'})$ are bounded in $L^1(\Omega)$, one has meas$(E_1) \leq \delta/3$, for t large enough, independently of ϵ, ϵ'. Thus we fix t in order to have

$$\text{meas } E_1 \leq \frac{\delta}{3}$$

We now take into account meas(E_3). Assumptions (2.2.4) implies that there exists a real valued function $\gamma(x)$ such that

$$\text{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0$$

and

$$[a(x, \xi) - a(x, \xi')] \cdot [\xi - \xi'] \geq \gamma(x),$$

for all $\zeta, \zeta' \in \mathbb{R}^N : |\zeta|, |\zeta'| \leq k, \ |\zeta - \zeta'| \geq t$, a.e. $x \in \Omega$. Indeed there exists a subset C of Ω such that meas$(C) = 0$ and the function $a(x, \zeta)$ is continuous with respect to ζ for any $x \in \Omega$. Then assumption (2.2.4) implies that for $x \in \Omega / C$ and $\zeta \neq \zeta'$ one has

$$(a(x, \zeta) - a(x, \zeta')) \cdot (\zeta - \zeta') > 0.$$

Define $K = \{(\zeta, \zeta') \in \mathbb{R}^{2N} : |\zeta| \leq k, \ |\zeta'| \leq k, \ |\zeta - \zeta'| \geq t\}$ Then

$$\inf \big\{((a(x, \zeta) - a(x, \zeta')) \cdot (\zeta - \zeta') : (\zeta, \zeta') \in K\big\} = \gamma(x) > 0, \tag{2.3.20}$$

since K is compact, in view of (2.3.20)

$$\int_{E_3} \gamma(x) dx \leq \int_{E_3} (a(x, \nabla u_\epsilon) - a(x, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) dx$$

if we use $T_k(u_\epsilon - u_{\epsilon'})$ in (2.3.15) as test function (where T_k is the usual function at level $\pm k$), we can say that the last integral is less or equal to $2kM$, where $M \geq \|\mu_{\epsilon,k}\|_{\mathcal{M}_b}$. Thus

$$\int_{E_3} \gamma(x) dx \leq \int_{E_3} (a(x, \nabla u_\epsilon) - a(x, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) dx \leq 2M\eta \tag{2.3.21}$$

by choosing $\eta = \frac{\delta}{3M}$. From Lemma 2.14 again, it follows that meas $E_3 < \frac{\delta}{3}$ independently of ϵ and ϵ'.
Now we fix such a k and thanks to the fact that u_ϵ is a Cauchy sequence in measure, we choose ϵ_0 such that

$$\text{meas } E_2 \leq \frac{\delta}{3} \quad \text{for } \epsilon, \epsilon' \geq \epsilon_0.$$

As a consequence, (∇u_ϵ) converges in measure to some measurable function v. Finally, since $(\nabla T_k(u_\epsilon))$ is bounded in $L^{p(\cdot)}(\Omega)$, for all $k>0$, it converges weakly to $\nabla T_k(u)$ in $L^1(\Omega)$. Therefore, v coincides with the weak gradient of u.

(iv) Using (2.2.3), we have

$$|a(x,\nabla u_\epsilon)| \leq \beta(j(x)+|\nabla u_\epsilon|^{p(x)-1}),$$

with $j \in L^{p'(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$, for all $1 \leq q(\cdot) < \frac{N}{N-1}$. We have that $(|\nabla u_\epsilon|^{p(\cdot)-1})$ is bounded in $L^{q(\cdot)}(\Omega)$. Hence, using Fatou's lemma, (2.2.3) and Vitali's theorem, we obtain that

$$\begin{aligned}
&|\nabla u|^{p(\cdot)-1} \in L^{q(\cdot)}(\Omega), \quad \forall\, q(\cdot) < \frac{N}{N-1},\\
&a(x,\nabla u_\epsilon) \to a(x,\nabla u) \text{ in } L^{q(\cdot)}(\Omega), \quad \forall\, 1 \leq q(\cdot) < \frac{N}{N-1}.
\end{aligned}$$

Since $a(x,\nabla T_k(u_\epsilon))$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ (By assumption (2.2.3) and (ii)) and by (iii), we have that it converges weakly to $a(x,\nabla T_k(u))$.

2.4. Proof of the main result

We now prove the main Theorem in this paper, we essentially follow the proof of **[DMOP]**, Theorem 3.4, and adapt it to the exponent case. Let

$$\mu = \mu_0 + \mu_s^+ - \mu_s^-, \quad \mu_\epsilon = \mu_{\epsilon,0} + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus,$$

be the decomposition of μ and μ_ϵ given by (2.1.1) and (2.3.8), and E^+, E^- be the disjoint sets where μ_s^+, μ_s^- are concentrated. Let u_ϵ be any solution of (2.3.14). By definition, if $w \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $r(\cdot) > N$, such that $w = w^+$ a.e. on the set $\{u_\epsilon > k\}$ and $w^{+\infty}$ a.e. on the set $\{u_\epsilon < -k\}$, then

$$\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla w dx = \int_\Omega w d\mu_{\epsilon,0} + \int_\Omega w^{+\infty} d\lambda_\epsilon^\oplus - \int_\Omega w^{-\infty} d\lambda_\epsilon^\ominus.$$

Using that the sequence (u_ϵ) and the function u are such that all the convergences considered in the previous Section hold. Hence we can pass to the limit on ϵ, proving that u is a distributional solution to (2.2.1), i.e. it solves

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\; dx = \int_\Omega \varphi d\mu, \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Now we want to prove that u is also a renormalized solution. First of all notice that u has the regularity results stated in (a) and (b) of Definition 2.6. Hence it remains to prove that it satisfies (c).

Let us now take $w = h(u_\epsilon)\varphi$ such that $h \in W^{1,\infty}(\mathbb{R})$ and $\varphi \in W_0^{1,r(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $r(\cdot) > N$, we have also that $w \in W_0^{1,p(\cdot)}(\Omega)$ (and hence $h(u)\varphi$ is an admissible test function in (2.2.5) for u replaced by u_ϵ). Recalling that if h' has compact support

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon h'(u_\epsilon)\varphi dx + \int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\varphi h(u_\epsilon)dx\\
&= \int_\Omega f_\epsilon h(u_\epsilon)\varphi dx - \langle \operatorname{div}(g_\epsilon), h(u_\epsilon)\varphi\rangle + \int_\Omega h(u_\epsilon)\varphi d\lambda_{\epsilon,0}^\oplus\\
&- \int_\Omega h(u_\epsilon)\varphi d\lambda_{\epsilon,0}^\ominus + h(+\infty)\int_\Omega \varphi d\mu_{\epsilon,s}^+ - h(-\infty)\int_\Omega \varphi d\mu_{\epsilon,s}^-.
\end{aligned} \tag{2.4.1}$$

In order to pass to the limit in the first term, we need the following improvement of (ii) of Proposition 2.13, since $\nabla T_k(u_\epsilon)$ converges to $\nabla T_k(u)$ strongly in $(L^{p(\cdot)}(\Omega))^N$, $\nabla u_\epsilon h'(u_\epsilon)$ converges to $\nabla u h'(u)$ weakly in $(L^{p(\cdot)}(\Omega))^N$ and φ belongs to $L^\infty(\Omega)$, we conclude that

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon h'(u_\epsilon)\varphi dx = \lim_{\epsilon\to0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)h'(u_\epsilon)\varphi dx\\
&= \lim_{\epsilon\to0}\int_\Omega a(x,\nabla T_k(u))\cdot\nabla T_k(u)h'(u)\varphi dx = \int_\Omega a(x,\nabla u)\cdot\nabla u h'(u)\varphi dx.
\end{aligned}$$

Furthermore, for the second term on the left-hand side of (2.4.1) we have by (iv) of Proposition 2.13, $a(x,\nabla u_\epsilon)$ converges to $a(x,\nabla u)$ strongly in $(L^{q(\cdot)}(\Omega))^N$, for every $1 \le q(\cdot) < \frac{N}{N-1}$ and $h(u_\epsilon)$ converges to $h(u)$ weakly–* in $L^\infty(\Omega)$, due to the fact that h belongs to $W^{1,\infty}(\mathbb{R})$, we can pass to the limit

$$\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\varphi h(u_\epsilon)dx = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi h(u)dx.$$

Concerning the right hand side, the first convergence is obvious since f_ϵ converges to f strongly in $L^1(\Omega)$, $h(u_\epsilon)\varphi$ converges to $h(u)\varphi$ weakly–* in $L^\infty(\Omega)$ and a.e. in Ω, and since $-\mathrm{div}(g_\epsilon)$ converges to $-\mathrm{div}(g)$ strongly in $W^{-1,p'(\cdot)}(\Omega)$, $h(u_\epsilon)\varphi$ converge to $h(u)\varphi$ weakly in $W_0^{1,p(\cdot)}(\Omega)$. Then we have

$$\lim_{\epsilon\to 0}\int_\Omega f_\epsilon h(u_\epsilon)\varphi - \langle \mathrm{div}(g_\epsilon), h(u_\epsilon)\varphi\rangle = \int_\Omega fh(u)\varphi - \langle \mathrm{div}(g), h(u)\varphi\rangle.$$

To conclude, let consider the last terms of (2.4.1), and for which it's enough to treat the sum $\int_\Omega h(u_\epsilon)\varphi d\lambda^\oplus_{\epsilon,0} + h(+\infty)\int_\Omega \varphi d\mu^+_{\epsilon,s}$. Setting $\nu_\epsilon = \lambda^\oplus_{\epsilon,s} - \mu^+_{\epsilon,s}$, we can write

$$\begin{aligned}&\int_\Omega h(u_\epsilon)\varphi d\lambda^\oplus_{\epsilon,0} + h(+\infty)\int_\Omega \varphi d\mu^+_{\epsilon,s}\\ &= \int_\Omega [h(u_\epsilon) - h(+\infty)]\varphi d\lambda^\oplus_{\epsilon,0} + h(+\infty)\int_\Omega \varphi d\lambda^\oplus_\epsilon - h(+\infty)\int_\Omega \varphi d\nu_\epsilon.\end{aligned}$$

By the fact that $0 \le \mu^+_{\epsilon,0} \le \lambda^\oplus_{\epsilon,0}$, and by the nonnegativity of $\lambda^\oplus_{\epsilon,0}$, for some $\nu_\epsilon \in \mathcal{M}_b(\Omega)$ with $0 \le \nu_\epsilon \le \lambda^\oplus_\epsilon = \lambda^\oplus_{\epsilon,0} + \lambda^\oplus_{\epsilon,s}$. Then there exist a subsequence, still denoted by (ν_ϵ), which converges in the narrow topology to a measure ν with $0 \le \nu \le \mu^+_s$. As $\mu_{\epsilon,s} = \lambda^\oplus_{\epsilon,s} - \lambda^\ominus_{\epsilon,s}$, we also have $\nu_\epsilon = \lambda^\ominus_{\epsilon,s} - \mu^-_{\epsilon,s}$, so that $0 \le \nu \le \mu^-_s$. Since μ^+_s and μ^-_s are mutually singular, we infer that $\nu = 0$, so that the whole sequence ν_ϵ converge to 0 in the narrow topology of measures, that is

$$\lim_{\epsilon\to 0}\int_\Omega \varphi d\nu_\epsilon = 0.$$

Now, if $\lambda^\oplus_\epsilon$ is as in the statement (2.3.11) we have, for every $\epsilon > 0$

$$\int_\Omega \varphi d\lambda^\oplus_\epsilon = \int_\Omega \varphi d\mu^+_s = \omega(\epsilon).$$

While recalling that if $\mathrm{supp}(h') \subseteq [-M, M]$, then $h(u_\epsilon) - h(+\infty) = 0$ on the set $\{u_\epsilon > M\}$, and so for the other term

$$\left|\int_\Omega [h(u_\epsilon) - h(+\infty)]\varphi d\lambda^\oplus_{\epsilon,0}\right| \le 2\|h\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\{u_\epsilon\le M\}} d\lambda^\oplus_{\epsilon,0}. \tag{2.4.2}$$

It remains to estimate $\lambda^\oplus_{\epsilon,0}(\{u_\epsilon \le M\})$ (and the analogous term $\lambda^\ominus_{\epsilon,0}(\{u_\epsilon > M\})$). First of all, we have to consider the cut-off functions ψ^+_δ and ψ^-_δ introduced in the following Lemma, proved in [**DMOP**], Lemma 6.1.

Lemma 2.15. *Let μ_s be a measure in $\mathcal{M}_s(\Omega)$, and let μ^+_s, μ^-_s be respectively the positive and negative part of μ_s. Then, for every $\delta > 0$, there exist two functions ψ^+_δ and ψ^-_δ in $C^\infty_c(\Omega)$, such that the following hold:*

1. $0 \le \psi^+_\delta \le 1$ *and* $0 \le \psi^-_\delta \le 1$ *on* Ω,
2. $\lim_{\delta\to 0}\psi^+_\delta = \lim_{\delta\to 0}\psi^-_\delta = 0$ *strongly in* $W_0^{1,p(\cdot)}(\Omega)$, *and weakly–* in* $L^\infty(\Omega)$,
3. $\int_\Omega \psi^-_\delta d\mu^+_s \le \delta$ *and* $\int_\Omega \psi^+_\delta d\mu^-_s \le \delta$,
4. $\int_\Omega (1 - \psi^+_\eta\psi^+_\delta)d\mu^+_s \le \delta + \eta$ *and* $\int_\Omega (1 - \psi^-_\eta\psi^-_\delta)d\mu^-_s \le \delta + \eta$ *for every* $\eta > 0$.

Lemma 2.16. *If $\lambda^\oplus_\epsilon$ and $\lambda^\ominus_\epsilon$ satisfy (2.3.11) and (2.3.12) respectively, and $\psi^-_\delta, \psi^+_\delta$ are the functions defined in Lemma 2.15, as an easy consequence of the narrow convergence, we obtain*

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_\Omega \psi^-_\delta d\lambda^\oplus_\epsilon = 0, \qquad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_\Omega \psi^+_\delta d\lambda^\ominus_\epsilon = 0, \tag{2.4.3}$$

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_\Omega (1 - \psi^+_\delta\psi^+_\eta)d\lambda^\oplus_\epsilon = 0, \qquad \lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_\Omega (1 - \psi^-_\delta\psi^-_\eta)d\lambda^\ominus_\epsilon = 0. \tag{2.4.4}$$

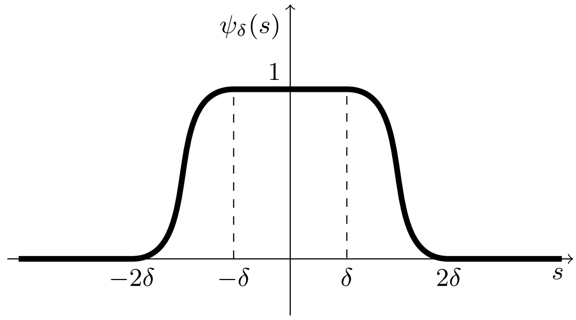


FIGURE 12. Example of cut-off functions

We want to stress that the use of doubly cut-off functions $\psi_\delta^+\psi_\eta^+$ was introduced essentially to control this terms. The following estimates will readily follow from Lemma 2.16 by a quite standard argument, we can write

$$\int_{\{u_\epsilon\le M\}} d\lambda_{\epsilon,0}^{\oplus} = \int_{\{u_\epsilon\le M\}} (1-\psi_\delta^+\psi_\eta^+)d\lambda_{\epsilon,0}^{\oplus} + \int_{\{u_\epsilon\le M\}} \psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus}.$$

So we have

$$0\le \int_{\{u_\epsilon\le M\}} (1-\psi_\delta^+\psi_\eta^+)d\lambda_{\epsilon,0}^{\oplus} \le \int_\Omega (1-\psi_\delta^+\psi_\eta^+)d\lambda_{\epsilon}^{\oplus},$$

which implies, thanks to Lemma 2.16, that

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{\{u_\epsilon\le M\}} (1-\psi_\delta^+\psi_\eta^+)d\lambda_{\epsilon}^{\oplus} = 0$$

Furthermore, for $k=M+1$ one has $0\le \chi_{\{-\infty,M\}}(t)\le k-T_k(t)$, for every $t\in\mathbb{R}$. Therefore we have, for $n>k$,

$$\begin{aligned}0\le \int_{\{u_\epsilon\le M\}} \psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus} &\le \int_\Omega (k-T_k(u_\epsilon))\psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus}\\ &\le \int_{\{-n\le u_\epsilon\le k\}} (k-T_k(u_\epsilon))\psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus} + 2k\int_{\{u_\epsilon<-n\}} \psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus}.\end{aligned}$$

To emphasize this interesting property, we need two technical Lemmas.

LEMMA 2.17. *Let k be a positive real number. Let f_ϵ, g_ϵ, $\lambda_\epsilon^{\oplus}$ and $\lambda_\epsilon^{\ominus}$ be sequences which satisfy* (2.3.9) – (2.3.12), *and let u_ϵ be a sequence of renormalized solution of* (2.3.14) *which satisfies* (i) – (ii) – (iii) *and* (iv) *of Proposition 2.13. For $\delta>0$ and $\eta>0$ given. Let $\psi_\delta^+,\psi_\delta^-$, and ψ_η^+,ψ_η^- be functions in $C_c^\infty(\Omega)$ which satisfy Lemmas 2.15 and 2.16. We then have*

$$\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\psi_\delta^+\psi_\eta^+dx = \omega(\eta,\delta,\epsilon),$$

$$\int_{\{-n\le u_\epsilon\le k\}} (k-T_k(u_\epsilon))\psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus} = \omega(\eta,n,\delta,\epsilon),$$

and

$$\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\psi_\delta^-\psi_\eta^-dx = \omega(\eta,\delta,\epsilon),$$

$$\int_{\{-k\le u_\epsilon\le n\}} (k-T_k(u_\epsilon))\psi_\delta^-\psi_\eta^- d\lambda_{\epsilon,0}^{\ominus} = \omega(\eta,n,\delta,\epsilon).$$

LEMMA 2.18. *Let $f_\epsilon, g_\epsilon, \lambda_\epsilon^{\oplus}$, and $\lambda_\epsilon^{\ominus}$ be sequences which satisfy* (2.3.9) – (2.3.12), *and let u_ϵ be a sequence of renormalized solution of* (2.3.14) *which satisfies* (i) – (ii) – (iii) *and* (iv) *of Proposition 2.13. Let η be a positive real number, and let $\Phi_\eta^{\oplus},\Phi_\eta^{\ominus}$ be functions in $W^{1,\infty}(\Omega)$ such that*

$$0\le\Phi_\eta^{\ominus}\le 1,\quad 0\le\Phi_\eta^{\oplus}\le 1,$$

$$0\le\int_\Omega \Phi_\eta^{\ominus}d\mu_s^+\le\eta,\quad 0\le\int_\Omega\Phi_\eta^{\oplus}d\mu_s^-\le\eta.$$

We then have

$$\frac{1}{n}\int_{\{n\le u_\epsilon<2n\}} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon \Phi_\eta^{\ominus} dx \le \omega_\eta(n,\epsilon)+\eta,$$
$$\int_{\{u_\epsilon>2n\}} \Phi_\eta^{\ominus} d\lambda_{\epsilon,0}^{\ominus} \le \omega_\eta(n,\epsilon)+\eta,$$
$$\frac{1}{n}\int_{\{-2n\le u_\epsilon<-n\}} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon \Phi_\eta^{\oplus} dx \le \omega_\eta(n,\epsilon)+\eta,$$
$$\int_{\{u_\epsilon<-2n\}} \Phi_\eta^{\oplus} d\lambda_{\epsilon,0}^{\oplus} \le \omega_\eta(n,\epsilon)+\eta.$$

Finally, thanks to Lemmas 2.17 and 2.18,

$$\lim_{\eta\to 0}\lim_{n\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{\{-n\le u_\epsilon\le k\}} (k-T_k(u_\epsilon))\psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus}=0, \tag{2.4.5}$$

$$\lim_{\eta\to 0}\lim_{n\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{\{u_\epsilon\le -n\}} \psi_\delta^+\psi_\eta^+ d\lambda_{\epsilon,0}^{\oplus}=0. \tag{2.4.6}$$

Hence we obtain

$$\lim_{\epsilon\to 0}\left|\int_\Omega (h(u_\epsilon)-h(+\infty))\varphi d\lambda_{\epsilon,0}^{\oplus}\right|=0. \tag{2.4.7}$$

Putting together (2.4.2) – (2.4.7), we have

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\lambda_{\epsilon,0}^{\oplus}+h(+\infty)\int_\Omega \varphi d\mu_{\epsilon,s}^+ = h(+\infty)\int_\Omega \varphi d\mu_s^+.$$

The estimate $\lambda_{\epsilon,0}^{\ominus}(\{u_\epsilon>M\})$ is obtained in the some way, choosing $k+T_k(u_\epsilon)$ and using the corresponding Lemmas 2.17 and 2.18, we have

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\lambda_{\epsilon,0}^{\ominus}-h(-\infty)\int_\Omega \varphi d\mu_{\epsilon,s}^- = h(-\infty)\int_\Omega \varphi d\mu_s^-.$$

And this concludes

$$\int_\Omega h(u)\varphi d\mu_0+h(+\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^-,$$

that is (2.2.8), as $\mu_0=f-\mathrm{div}(g)$, which implies Theorem 2.9. □

CHAPTER 3

Nonlinear parabolic problems with diffuse measure data and variable exponent

A large number of papers was devoted to the study of solutions for parabolic problems under various assumptions, for elliptic problems the reader should consult Chapter 2 for more details, for a review on classical parabolic results, see [**B, BG1, DL1, L**] and references therein. In [**AS, AZ, YL**] some anisotropic problems with variable exponents are studied and in [**AAR, El**] for the framework of weight Sobolev spaces and Orlicz spaces. Moreover, in the case when μ belongs to the dual of the parabolic Sobolev spaces, we refer to [**L**], see also [**BM, Pr2, AMST**] for L^1-data. General results for a finite Radon measure can be found in [**BGO1, DPP, P**], another approaches can be found in [**PPP1, PPP2**] for diffuse measures and in [**Pe1, Pe3**] for singular measures. More recently in [**AHT, YAR, ABR**] for a class of problems different to the one we will discuss. Actually we shall investigate the relationship between parabolic $p(\cdot)-$capacity and diffuse measures. Observe that by virtue of decomposition result of Lemma 1.43 we have $\mu = f - \mathrm{div}(G) + g_t$, where $f \in L^1(Q)$, $G \in L^{p'}(Q)$ and $g \in L^p(0,T;V)$ with $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, so the decomposition is well defined for all t, we are interested in the extension of this decomposition result in exponent case. Actually, for a larger class of measures we shall prove that, as μ is decomposed in space and time, the renormalized solution of the corresponding parabolic problem with μ as data exist and is unique. The main technical tools used include estimates and compactness convergences. This Chapter is organized as follows. In Section 3.1, we recall some basic properties on functional spaces and $p(\cdot)-$parabolic capacity. In section 3.2 and 3.3, we state the precise hypotheses on the data and the main result. We then quickly prove some a priori estimates and properties of renormalized solutions. Finally, in Section 3.4, we show how these estimates allow to obtain existence of solutions. Our argument will be based on a special type of distributional solutions, the so-called "renormalized solutions" and also on the strong convergence of truncates.

3.1. Parabolic $p(\cdot)-$capacity and diffuse measures

In this part, we shall mainly work with capacities of compact sets, since we are interested in local properties, we restrict our attention to $U \subset Q$, where U is an open set. Then, we begin with a general definition (in the same spirit of Pierre [**P**]) of the space $W_{p(\cdot)}(0,T)$ and the parabolic $p(\cdot)-$capacity.

Definition 3.1. Let us define $V = W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$ endowed with its natural norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and the space

$$W_{p(\cdot)}(0,T) = \{u \in L^{p_-}(0,T;V);\ \nabla u \in L^{p(\cdot)}(Q),\ u_t \in L^{p'_-}(0,T;V')\}$$

endowed with its natural norm

$$\|u\|_{W_{p(\cdot)}(0,T)} = \|u\|_{L^{p_-}(0,T;V)} + \|\nabla u\|_{L^{p(\cdot)}(Q)} + \|u_t\|_{L^{p'_-}(0,T;V')}.$$

Definition 3.2. The parabolic $p(\cdot)-$capacity of an arbitrary subset E of Q is

$$\mathrm{cap}_{p(\cdot)}(E) = \inf \{\|u\|_{W_{p(\cdot)}(0,T)};\ u \in W_{p(\cdot)}(0,T),\ u > \chi_U \text{ a.e. in } Q\}. \tag{3.1.1}$$

If the set, over which the infimum is taken, is not bounded from above, then we set $\mathrm{cap}_{p(\cdot)}(E) = 0$.

Remark 3.3. Notice also that. The parabolic capacity can be expressed in terms of Borelian subset as

$$\mathrm{cap}_{p(\cdot)}(B) = \inf \{\mathrm{cap}_{p(\cdot)}(U),\ U \text{ open subset of } Q,\ B \subset U\}. \tag{3.1.2}$$

It also follows immediately from the definition that if $E_1 \subset E_2$, then

$$\mathrm{cap}_{p(\cdot)}(E_1) \leq \mathrm{cap}_{p(\cdot)}(E_2). \tag{3.1.3}$$

Thus, the parabolic capacity is a monotonic set function. And for $E_i,\ i \in \mathbb{N}$ be arbitrary subsets of Q and $E = \cup_{i=1}^{\infty} E_i$. Then,

$$\mathrm{cap}_{p(\cdot)}(E) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{p(\cdot)}(E_i). \tag{3.1.4}$$

The parabolic capacity is also countably sub-additive.

The next result shows that the capacity is inner regular

LEMMA 3.4. *Let Ω be a bounded subset of $\mathbb{R}^N$ and $1 < p_- < p_+ < \infty$. Then $C_c^{\infty}([0,T] \times \Omega)$ is dense in $W_{p(\cdot)}(0,T)$.*

PROOF. See **[OT]**, Proposition 3.3. □

DEFINITION 3.5. Let K be a compact subset of Q. the capacity of K is defined as

$$\mathrm{cap}_{p(\cdot)}(K) = \inf\ \{\|u\|_{W_{p(\cdot)}(0,T)} :\ u \in C_c^{\infty}([0,T] \times \Omega),\ u > \chi_K\}.$$

The capacity of any open subset U of Q is then defined by

$$\mathrm{cap}_{p(\cdot)}(U) = \sup\ \{\mathrm{cap}_{p(\cdot)}(K),\ K \text{ compact},\ K \subset U\}$$

and the capacity of any Borelian set $B \subset Q$ by

$$\mathrm{cap}_{p(\cdot)}(B) = \inf\ \{\mathrm{cap}_{p(\cdot)}(U),\ U \text{ open subset of } Q,\ B \subset U\}.$$

DEFINITION 3.6. A claim is said to hold $\mathrm{cap}_{p(\cdot)}$ quasi-everywhere if it holds everywhere, except on a set of zero $p(\cdot)-$capacity. A function $u : Q \to \mathbb{R}$ is said to be $\mathrm{cap}_{p(\cdot)}$ quasi-continuous if for $\epsilon > 0$, there exists an open set U_ϵ with $\mathrm{cap}_{p(\cdot)}(U_\epsilon) < \epsilon$ such that u restricted to $Q\backslash U_\epsilon$ is continuous.

In fact, the natural space that appears in the study of nonlinear parabolic operators is not $W_{p(\cdot)}(0,T)$ but $\overline{W}_{p(\cdot)}(0,T) \subset W_{p(\cdot)}(0,T)$. Following the outlines of **[OT]**, let us also define $\overline{W}_{p(\cdot)}(0,T)$ by

$$\begin{aligned}\overline{W}_{p(\cdot)}(0,T) = \{&u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \cap L^{\infty}(0,T; L^2(\Omega));\ \nabla u \in (L^{p(\cdot)}(Q))^N,\\ &u_t \in L^{p'_-}(0,T; W^{-1,p'(\cdot)}(\Omega))\}\end{aligned}$$

and for all $z \in \overline{W}_{p(\cdot)}(0,T)$, let us denote

$$[z]_{W_{p(\cdot)}(0,T)} = \|z\|^{p_-}_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|z_t\|^{p'_-}_{L^{p'_-}(0,T;V')} + \|z\|^2_{L^{\infty}(0,T;L^2(\Omega))}.$$

In **[OT]**, the authors has shown the following result that we present in this Chapter as a Lemma. For the sake of simplicity, we use the notations

$$\begin{aligned}[u]_* = {}&\rho_{p(\cdot)}(|\nabla u|) + \|u_t\|^2_{L^{(p')_-}(0,T;V')} + \|u\|^2_{L^{\infty}(0,T;L^2(\Omega))} + \|u_t\|^{p'_-}_{L^{p'_-}(0,T;V')}\\ &+ \|u_t\|_{L^{p'_-}(0,T;V')} + \|u_t\|_{L^{p'_-}(0,T;V')}\|u\|_{L^{\infty}(0,T;L^2(\Omega))}\end{aligned}$$

and

$$\begin{aligned}[u]_{**} = {}&\rho_{p(\cdot)}(|\nabla u|) + \|u\|^2_{L^{\infty}(0,T;L^2(\Omega))} + \|u_t\|^{p'_-}_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)}\\ &+ \|u_t\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)} + \|u_t\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)}\|u\|_{L^{\infty}(Q)}.\end{aligned}$$

LEMMA 3.7. *Let $u \in W_{p(\cdot)}(0,T)$, then there exists $z \in \overline{W}_{p(\cdot)}(0,T)$ such that $|u| \leq z$ and*

$$[z]_{W_{p(\cdot)}} \leq C([u]_{**} + [u]_{**}^{\frac{1}{p_-}} + [u]_{**}^{\frac{1}{p_+}} + [u]_{**}^{\frac{1}{(p')_-}} + [u]^{\frac{1}{(p')_+}}),$$

where $u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \cap L^{\infty}(Q),\ u_t \in L^{p_-}(0,T; W^{-1,p'(\cdot)}(\Omega)) + L^1(Q)$ and

$$\|z\|_{\overline{W}_{p(\cdot)}(0,T)} \leq C([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}).$$

Now our aim is to prove the following result

THEOREM 3.8. *Let $u \in W_{p(\cdot)}(0,T)$; then u admits a unique cap$_{p(\cdot)}$ quasi-continuous representative defined cap$_{p(\cdot)}$ quasi-everywhere.*

To prove Theorem 3.8, we need first a capacitary estimate, that is the goal of the following result.

LEMMA 3.9. *Let $u \in W_{p(\cdot)}(0,T)$ be cap$_{p(\cdot)}$ quasi-continuous, then for every $k > 0$,*

$$cap_{p(\cdot)}(\{|u| > k\}) \leq \frac{c}{k}\max(\|u\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}). \tag{3.1.5}$$

PROOF. See **[OT]**, Proposition 3.16. □

Proof of Theorem 3.8. Let us first observe that we can approximate a function $u \in W_{p(\cdot)}(0,T)$ with smooth functions $u^m \in C_0^\infty([0,T] \times \Omega)$ in the norm of $W_{p(\cdot)}(0,T)$ using convolution arguments; so let u^m be a sequence such that

$$\sum_{m=1}^{\infty} 2^m \max\{\|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}\} \text{ is finite.}$$

For every m and r, let us define

$$\omega^m = \{|u^{m+1} - u^m| > \frac{1}{2^m}\} \text{ and } \Omega^r = \underset{m \geq r}{\cup} \omega^m.$$

Now we can apply Lemma 3.9 to obtain

$$\text{cap}_{p(\cdot)}(\omega^m) \leq C\, 2^m \max\{\|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}\}$$

and so, by sub-additivity,

$$\text{cap}_{p(\cdot)}(\Omega^r) \leq C \sum_{m \geq r} 2^m \max\{\|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}\};$$

which implies that

$$\lim_{r \to \infty} \text{cap}_{p(\cdot)}(\Omega^r) = 0. \tag{3.1.6}$$

Moreover, for every $y \notin \Omega^r$ we have

$$|u^{m+1} - u^m|(y) \leq \frac{1}{2^m}.$$

For any $m \geq r$, u^m converges uniformly on the complement of Ω^r and pointwise on the complement of $\cap_{r=1}^\infty \Omega^r$. But, for any $l \in \mathbb{N}$, we have

$$\text{cap}_{p(\cdot)}(\cap_{r=1}^\infty \Omega^r) \leq \text{cap}_{p(\cdot)}(\Omega^l),$$

and so, by (3.1.6), we conclude that $\text{cap}_{p(\cdot)}(\cap_{r=1}^\infty \Omega^r) = 0$; therefore the limit of u^m is cap$_{p(\cdot)}$ quasi-continuous and is defined cap$_{p(\cdot)}$ quasi-everywhere. Let us denote $\tilde{u}$ this cap$_{p(\cdot)}$ quasi-continuous representative of u, and let z be another cap$_{p(\cdot)}$ quasi-continuous representative of u; thanks to Lemma 3.9, for any $\epsilon > 0$, we have

$$\text{cap}_{p(\cdot)}(\{|\tilde{u} - z| > \epsilon\}) \leq \frac{C}{\epsilon}(\|\tilde{u} - z\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|\tilde{u} - z\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}) = 0,$$

since $\tilde{u} = z$ in $W_{p(\cdot)}(0,T)$ and this conclude the proof.

Now, as in Section 1.12, denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the $\sigma-$algebra of Borelian of Q, and by $\mathcal{M}_b^+(Q)$ the subsets of nonnegative measures of $\mathcal{M}_b(Q)$, with the symbol $\mathcal{M}_0(Q)$ we mean a measure with bounded variation over Q which does not charge the sets of zero $p(\cdot)-$capacity, this measures μ are called soft or diffuse measures. We refer the reader to **[OT]** for further specifications about parabolic $p(\cdot)-$capacity. Let us define the space $\mathcal{M}_0(Q)$ as

DEFINITION 3.10. Let E be a subset of Q. the space $\mathcal{M}_0(Q)$ is defined as

$$\mathcal{M}_0(Q) = \{\mu \in \mathcal{M}_b(Q) : \ \mu(E) = 0, \ \forall E \subset Q \text{ such that } \text{cap}_{p(\cdot)}(E) = 0\}.$$

We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $W'_{p(\cdot)}(0,T)$ and $W_{p(\cdot)}(0,T)$, if $\gamma\in W'_{p(\cdot)}(0,T)$ such that there exists $c>0$ satisfying $\langle\gamma,\varphi\rangle\le C\|\varphi\|_{L^\infty(Q)}$ for every function $\varphi\in C_c^\infty(Q)$. Then, $\gamma\in W'_{p(\cdot)}(0,T)\cap\mathcal{M}_b(Q)$ and is identified by unique linear application $\varphi\in C_c^\infty(Q)\to\int_Q\varphi\gamma^{\mathrm{meas}}$ when γ^{meas} belongs to $\mathcal{M}_b(Q)$. This shows that we need to detail the structure of the dual space $W'_{p(\cdot)}(0,T)$.

LEMMA 3.11. *Let $g\in W'_{p(\cdot)}(0,T)$, then there exists $g_1\in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$, $g_2\in L^{p_-}(0,T;V)$, $F\in(L^{p'(\cdot)}(Q))^N$ and $g_3\in L^{p'_-}(0,T;L^2(\Omega))$ such that*

$$\langle g,u\rangle=\int_0^T\langle g_1,u\rangle dt-\int_0^T\langle u_t,g_2\rangle+\int_Q F\cdot\nabla u\;dxdt+\int_Q g_3u\;dxdt,\ \forall u\in W_{p(\cdot)}(0,T)$$

and there exist a constant C (do not depend on g) such that

$$\|g_1\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))}+\|g_2\|_{L^{p_-}(0,T;V)}+\|F\|_{L^{p'(\cdot)}(Q)}+\|g_3\|_{L^{p'_-}(0,T;L^2(\Omega))}\le C\,\|g\|_{W'_{p(\cdot)}(0,T)}.$$

PROOF. See **[OT]**, Lemma 4.2. □

The next Lemma will play an essential role in this context.

LEMMA 3.12. *Let $\mu\in\mathcal{M}_0(Q)$, there exists a decomposition (g,h) of μ such that $g\in W'_{p(\cdot)}(0,T)$, $h\in L^1(Q)$ and*

$$\int_Q\varphi\;d\mu=\langle g,\varphi\rangle+\int_Q h\varphi\;dxdt\quad\forall\varphi\in C_c^\infty([0,T]\times\Omega).\tag{3.1.7}$$

PROOF. See **[OT]**, Lemma 4.4. □

Finally, the essential tool in this chapter is the following result.

THEOREM 3.13. *Let $\mu\in\mathcal{M}_0(Q)$, there exists a decomposition (f,F,g_1,g_2) of μ such that $f\in L^1(Q)$, $F\in(L^{p'(\cdot)}(Q))^N$, $g_1\in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$ and $g_2\in L^{p_-}(0,T;V)$ such that*

$$\int_Q\varphi\;d\mu=\int_Q f\varphi\;dxdt+\int_Q F\cdot\nabla\varphi\;dxdt+\int_0^T\langle g_1,\varphi\rangle dt-\int_0^T\langle\varphi_t,g_2\rangle dt,\quad\forall\varphi\in C_c^\infty([0,T]\times\Omega).$$

PROOF. The proof is a combination of the proofs of Lemma 3.11 and Lemma 3.12. □

REMARK 3.14. In general, the decomposition in $\mathcal{M}_0(Q)$ is not unique.

Indeed, we have the following result

LEMMA 3.15. *Let $\mu\in\mathcal{M}_0(Q)$ and let (f,F,g_1,g_2), $(\tilde f,\tilde F,\tilde g_1,\tilde g_2)$ be two different decompositions of μ according to Theorem 3.13. Then, we have*

$$\int_0^T\langle(g_2-\tilde g_2)_t,\varphi\rangle dt=\int_Q(\tilde f-f)\varphi\;dxdt+\int_Q(\tilde F-F)\cdot\nabla\varphi\;dxdt+\int_0^T\langle\tilde g_1-g_1,\varphi\rangle dt,\tag{3.1.8}$$

where $\varphi\in C_c^\infty([0,T]\times\Omega)$ and $g_2-\tilde g_2\in C([0,T];L^1(Q))$ with $(g_2-\tilde g_2)(0)=0$.

PROOF. See **[OT]**, Lemma 4.6. □

3.2. General assumptions and weak solutions

Throughout this chapter, we assume that Ω is a bounded open set on $\mathbb{R}^N$, $N\ge 2$, $Q=(0,T)\times\Omega$ and $a:Q\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (i.e. $a(\cdot,\cdot,\zeta)$ is measurable on Ω, for all $\zeta\in\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $(t,x)\in Q$ such that the following holds.

$$a(t,x,\zeta)\cdot\zeta\ge\alpha|\zeta|^{p(x)},\tag{3.2.1}$$

$$|a(t,x,\xi)|\le\beta[b(t,x)+|\zeta|^{p(x)-1}],\tag{3.2.2}$$

$$(a(t,x,\zeta)-a(t,x,\eta))\cdot(\zeta-\eta)>0,\tag{3.2.3}$$

for almost every $(t,x)\in Q$, for all $\zeta,\eta\in\mathbb{R}^N$ with $\zeta\neq\eta$, where $p_->1$, α,β are positive constants and b is a nonnegative function in $L^{p'(x)}(\Omega)$. For every $u\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ with $|\nabla u|\in(L^{p(\cdot)}(Q))^N$, let us define

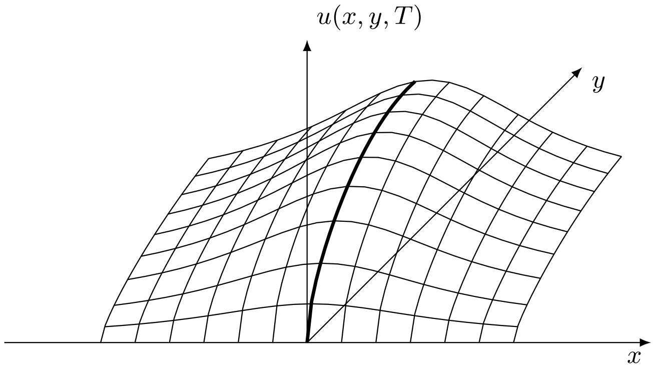


FIGURE 13. Example of solutions in $(0,T) \times \mathbb{R}^2$

the differential operator $A(u) = -\mathrm{div}(a(t,x,\nabla u))$, which, thanks to the assumptions on a, turns out to be a coercive monotone operator acting from the space $L^{p_-}(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$. We shall deal with the solutions of the initial boundary-value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{3.2.4}$$

where μ is a measure with bounded variation over $Q = (0,T)\times\Omega$, and $u_0 \in L^1(\Omega)$.
Let us fix $T > 0$. If $\mu \in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$, it is well known that problem (3.2.4) has a unique variational solution in $Q = (0,T)\times\Omega$ such that $u \in W_{p(\cdot)}(0,T) \cap C([0,T];L^2(\Omega))$, that is

$$\begin{aligned} &\int_0^T \langle u_t, \varphi\rangle_{W^{-1,p'(\cdot)}(\Omega),W_0^{1,p(\cdot)}(\Omega)}\,dt + \int_{Q_T} a(t,x,\nabla u)\cdot\nabla\varphi\,dx\,dt \\ &= \int_0^T \langle \mu, \varphi\rangle_{W^{-1,p'(\cdot)}(\Omega),W_0^{1,p(\cdot)}(\Omega)}\,dt. \end{aligned} \tag{3.2.5}$$

Then we mean that u is a weak solution of problem (3.2.4) if $u \in L^{p_-}(0,T;V)$, $|\nabla u| \in L^{p(\cdot)}(Q)$ and if

$$-\int_Q \langle\varphi_t, u\rangle dt - \int_\Omega u_0\varphi(0)\,dx + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi\,dxdt = \langle g, \varphi\rangle,$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$. Since we are going to deal with measures, the solution we will find will not belong in general to Sobolev spaces. For this reason, we are going to justify the interest of $W'_{p(\cdot)}(0,T)$, giving the following existence and uniqueness theorem.

THEOREM 3.16. *Let g belong to $W'_{p(\cdot)}(0,T)$, and let $u_0 \in L^2(\Omega)$. Then there exists a unique solution $u \in L^{p_-}(0,T;V)$ of (3.2.4) such that*

$$-\int_Q \langle\varphi_t, u\rangle dt - \int_\Omega u_0\varphi(0)\,dx + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi\,dxdt = \langle g, \varphi\rangle, \tag{3.2.6}$$

for all $\varphi \in W_{p(\cdot)}(0,T)$ with $\varphi(T) = 0$.

REMARK 3.17. Since $g \in W'_{p(\cdot)}(0,T)$, by Lemma 3.11 and (3.2.6), we deduce that u satisfies

$$(u-g_2)_t = -Au + g_1 - \mathrm{div}(F) + g_3 \in\ L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega)) + L^{p'_-}(0,T;L^2(\Omega)) = L^{p'_-}(0,T;V')).$$

Therefore, $u - g_2 \in W_{p(\cdot)}(0,T) \subset C([0,T];L^2(\Omega))$. Then by (3.2.6), $(u-g_2)(0) = u_0$. Moreover, for any two solutions u and v of (3.2.6), we have $u - v = u - g_2 - (v - g_2) \in W_{p(\cdot)}(0,T)$ and $(u-v)(0) = 0$.

REMARK 3.18. Theorem 3.16 could also be stated with right-hand side in $\overline{W}'_{p(\cdot)}(0,T)$ and test functions in $\overline{W}_{p(\cdot)}(0,T)$. Moreover, one has

$$\overline{W}_{p(\cdot)}(0,T) = \{u \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^2(0,T;L^2(\Omega)),\ |\nabla u| \in (L^{p(\cdot)}(Q))^N;\\ u_t \in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))\},$$

hence the right hand side $g_2 \in \overline{W}'_{p(\cdot)}(0,T)$ with $g_2 \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \supset L^{p_-}(0,T;V)$, the term $\int_0^T \langle \varphi_t, g_2\rangle$ makes sense also when $\varphi \in \overline{W}_{p(\cdot)}(0,T)$.

We will argue by density for proving the existence of solutions, so that we need the following preliminary result that applies for equations to obtain additional regularity on the renormalized solutions.

PROPOSITION 3.19. *Let $\mu \in \mathcal{M}_0(Q)$. Then there exists a decomposition (f, F, g_1, g_2) of μ in the sense of Theorem 3.13 and an approximation $\mu^\epsilon \in C_c^\infty(Q)$ satisfying $\|\mu^\epsilon\|_{L^1(Q)} \le C$ such that*

$$\int_Q \mu^\epsilon \varphi\ dxdt = \int_Q \varphi f^\epsilon\ dxdt + \int_Q F^\epsilon \nabla\varphi\ dxdt + \int_0^t \langle \operatorname{div} G_1^\epsilon, \varphi\rangle dt\\ - \int_0^t \langle \varphi, g_2^\epsilon\rangle dt, \quad \forall \varphi \in C_c^\infty([0,T]\times\Omega),$$

with (C not depending on ϵ)

$$\begin{cases} f^\epsilon \in C_c^\infty(Q) & \text{such that} \quad \|f^\epsilon - f\|_{L^1(Q)} \le C\epsilon, \\ F^\epsilon \in (C_c^\infty(Q))^N & \text{such that} \quad \|F^\epsilon - F\|_{(L^{p'(\cdot)}(Q))^N} \le C\epsilon, \\ G_1^\epsilon \in (C_c^\infty(Q))^N & \text{such that} \quad \|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \le C\epsilon, \\ g_2^\epsilon \in C_c^\infty(Q) & \text{such that} \quad \|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \le C\epsilon. \end{cases}$$

PROOF. From Definition 3.10, there exists $\gamma \in W'_{p(\cdot)}(0,T) \cap \mathcal{M}_b^+(\Omega)$ and a nonnegative Borel function $f \in C^1(Q, d\gamma^{\text{meas}})$ such that $\mu(B) = \int_B f d\gamma^{\text{meas}}$ for Borel set B in Q. From the fact that $C_c^\infty(Q)$ is dense in $L^1(Q, d\gamma^{\text{meas}})$, since γ^{meas} is a regular measure; there exists a sequence $f_n \in C_c^\infty(Q)$ such that f_n strongly converges to f in $L^1(Q, d\gamma^{\text{meas}})$. Then we can assume $\sum_{n=0}^\infty \|f_n - f_{n-1}\|_{L^1(Q,d\gamma^{\text{meas}})} < \infty$, and we define $\nu_n = (f_n - f_{n-1})\gamma \in W'_{p(\cdot)}(0,T)$, we have $\nu_n \in W'_{p(\cdot)}(0,T) \cap \mathcal{M}_b(Q)$ and $\sum_{n=0}^\infty \nu_n^{\text{meas}} = \sum_{n=0}^\infty (f_n - f_{n-1})\gamma^{\text{meas}}$ $= \mu$ converges in the strong topology of measures, $\rho_l * \nu_n^{\text{meas}}$ strongly converges to ν_n in $W'_{p(\cdot)}(0,T)$ as l tends to infinity, we can then extract a subsequence l_n such that $\|\rho_{l_n} * \nu_n^{\text{meas}} - \nu_n\|_{W'_{p(\cdot)}(0,T)} \le \frac{1}{2^n}$. We have then

$$\sum_{k=0}^n \nu_k^{\text{meas}} = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}} + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}).$$

Let us denote

$$m_n = \sum_{k=0}^n \nu_k^{\text{meas}},\ h_n = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}},\ g_n = \sum_{k=0}^n (\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$$

and $g_n^{\text{meas}} = \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})$. We have that h_n strongly converges in $L^1(Q)$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}}\|_{L^1(Q)} \le \|\nu_k^{\text{meas}}\|_{\mathcal{M}_b(Q)}$) and $\sum_{k=0}^\infty \nu_k^{\text{meas}}$ is totally convergent in $\mathcal{M}_b(Q)$, we denote by h its limite, we also have g_n is strongly convergent in $W'_{p(\cdot)}(0,T)$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}} - \nu_k\|_{W'_{p(\cdot)}(0,T)} \le \frac{1}{2^k}$), denoting by g its limit. Now, we choose $\zeta_k \in C_c^\infty(Q)$ such that $\zeta_k \equiv 1$ on a neighborhood of $\operatorname{supp}(f_n - f_{n-1})$; then there exists $C(\zeta_k)$ only depending on ζ_k such that

$$\begin{cases} \|\zeta_k h\|_E \le C(\zeta_k)\|h\|_E \text{ if } E \subset \{(L^{p'(\cdot)}(Q))^N, L^{p'_-}(0,T;V), L^{p'_-}(0,T;L^2(\Omega))\} \text{ and } h \in E; \\ \|H\nabla\zeta_k\|_{L^{p'(\cdot)}(Q)} \le C(\zeta_k)\|H\|_{(L^{p'(\cdot)})^N} \text{ if } H \in (L^{p'(\cdot)}(Q))^N; \\ \|(\zeta_k)_t h\|_{L^{p_-}(0,T;L^2(\Omega))} \le C(\zeta_k)\|h\|_{L^{p_-}(0,T;L^2(\Omega))} \text{ if } h \in L^{p_-}(0,T;L^2(\Omega)). \end{cases}$$

We choose l_k such that $\|\rho_{l_k} * \nu_n^{meas} - \nu_k\|_{W'_{p(\cdot)}(0,T)} \le \frac{1}{(2^k(C(\zeta_k)+1))}$ and $\zeta_k \equiv 1$ on a neighborhood of $\operatorname{supp}(\rho_{l_k} * \nu_k^{meas})$. Thanks to this choice and the decomposition $(b_0^k, \operatorname{div}(B_1^k), b_2^k, b_3^k)$ of $\nu_k - \rho_{l_k} * \nu_k^{meas}$, there exists a constant C (C not depending on k) such that

$$\|b_0^k\|_{(L^{p'(\cdot)}(Q))^N} + \|B_1^k\|_{(L^{p'(\cdot)}(Q))^N} + \|b_2^k\|_{L^{p_-}(0,T;V)} + \|b_3^k\|_{L^{p'_-}(0,T;L^2(\Omega))} \leq C\|\nu_k - \rho_{l_k} * \nu_k^{meas}\|_{W'_{p(\cdot)}(0,T)}.$$

So that we can write

$$(3.2.7) \qquad \begin{cases} \sum_{k\geq 1} \zeta_k b_0^k \text{ converges in } (L^{p'(\cdot)}(Q))^N, \ \sum_{k\geq 1} \zeta_k B_1^k \text{ converges in } (L^{p'(\cdot)}(Q))^N, \\ \sum_{k\geq 1} \zeta_k b_2^k \text{ converges in } L^{p_-}(0,T;V), \ \sum_{k\geq 1} \zeta_k b_3^k \text{ converges in } L^{p'_-}(0,T;L^2(\Omega)), \\ \sum_{k\geq 1} b_0^k \nabla\zeta_k \text{ converges in } L^{p'(\cdot)}(Q), \ \sum_{k\geq 1} B_1^k \nabla\zeta_k \text{ converges in } L^{p'(\cdot)}(Q), \\ \sum_{k\geq 1} (\zeta_k)_t b_2^k \text{ converges in } L^{p_-}(0,T;L^2(\Omega)). \end{cases}$$

We denote by $F_0, G, -g_2, f_0, f_1, f_2$ and f_3 the respective limits of the terms above; (3.2.7) imply the convergence in $L^1(Q)$. Since $\nu_k - \rho_{l_k} * \nu_k^{meas} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{meas})$ in $W'_{p(\cdot)}(0,T)$ and thanks to the choice of ζ_k and ρ_k and the decomposition $(b_0^k, \mathrm{div}(B_1^k), b_2^k, b_3^k)$ of $\nu_k - \rho_{l_k} * \nu_\epsilon^{meas}$, the last term admits a pseudo-decomposition $(\zeta_k b_0^k, \zeta_k B_1^k, \zeta_k b_2^k, \zeta_k b_3^k, -b_0^k\nabla\zeta_k, -B_1^k, (\zeta_k)_t b_2^k)$. Thus, as

$$\int_Q \varphi \ dm_n = \int_Q h_n\varphi \ dxdt + \langle g_n, \varphi\rangle,$$

we can write for all $\varphi \in C_c^\infty([0,T]\times\Omega)$,

$$\int_Q \varphi dm_n = \int_Q \varphi h_n + \int_0^t \langle \mathrm{div}(\sum_{k=0}^n \zeta_k b_0^k), \varphi\rangle + \int_0^t \langle \mathrm{div}(\sum_{k=0}^n \zeta_k B_1^k), \varphi\rangle + \int_0^t \langle \varphi_t, \sum_{k=0}^n \zeta_k b_2^k\rangle$$
$$+ \int_0^t \sum_{k=0}^n \zeta_k b_3^k\varphi + \int_Q \sum_{k=0}^n (-F_0^k\nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (-B_1^k\nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (\zeta_k)_t b_2^k\varphi.$$

From the convergences of m_n to μ, of h_n to h and using (3.2.7), we have

$$\int_Q \varphi d\mu = \int_Q (h + f_0 + f_1 - f_2 + f_3)\varphi + \int_0^t F\nabla\varphi + \int_0^t \langle \mathrm{div}(G), \varphi\rangle - \int_0^T (\varphi_t, g_2).$$

That is $(f = h + f_0 + f_1 - f_2 + f_3, F, \mathrm{div}(G), g_2)$ is a decomposition of μ in the sense of Theorem 3.13. Taking n large enough and $\epsilon > 0$ fixed, we obtain

$$(3.2.8) \qquad \begin{cases} \|\sum_{k=0}^n \zeta_k b_0^k - F\|_{(L^{p'(\cdot)}(Q))^N} \leq \epsilon, \\ \|\sum_{k=0}^n \zeta_k B_1^k - G_1\|_{(L^{p'(\cdot)}(Q))^N} \leq \epsilon, \\ \|\sum_{k=0}^n \zeta_k b_2^k + g_2\|_{L^{p_-}(0,T;V)} \leq \epsilon, \\ \|h_n + \sum_{k=0}^n \zeta_k b_3^k - \sum_{k=0}^n (b_0^k\nabla\zeta_k) - \sum_{k=0}^n (b_1^k\nabla\zeta_k) + \sum_{k=0}^n (\zeta)_t b_2^k - f\|_{L^1(Q)} \leq \epsilon. \end{cases}$$

Note that $\nu_k - \rho_{l_k} * \nu_k^{meas} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{meas})$ and $(b_0^k, \mathrm{div}(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{meas}$, note also that, for j large enough, $((\zeta_k b_0^k) * \rho_j, (\zeta_k B_1^k) * \rho_j, (\zeta_k b_2^k) * \rho_j, (\zeta_k b_3^k) * \rho_j, (-f_0^k\nabla\zeta_k) * \rho_j, ((\zeta_k)_t b_2^k) * \rho_j)$

is a pseudo decomposition of $(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_j \in C_c^\infty(Q)$. We take j_n such that, for all $k \in [0, n]$,

$$(3.2.9)\qquad \begin{cases} \|(\zeta_k b_0^k) * \rho_{j_n} - \zeta_k b_0^k\|_{(L^{p'(\cdot)}(Q))^N} \le \dfrac{\epsilon}{n+1}, \\ \|(\zeta_k B_1^k) * \rho_{j_n}\| - \zeta_k B_1^k\|_{(L^{p'(\cdot)}(Q))^N} \le \dfrac{\epsilon}{n+1}, \\ \|(\zeta_k b_2^k) * \rho_{j_n} - \zeta_k b_2^k\|_{L^{p_-}(0,T;V)} \le \dfrac{\epsilon}{n+1}, \\ \|(\zeta_k b_3^k) * \rho_{j_n} - \zeta_k b_3^k\|_{L^1(Q)} + \|(b_0^k \nabla\zeta_k) * \rho_{j_n} - b_0^k \nabla\zeta_k\|_{L^1(Q)} \\ \quad + \|(B_1^k \nabla\zeta_k) * \rho_{j_n} - B_1^k \nabla\zeta_k\|_{L^1(Q)} + \|(\zeta_k)_t b_2^k) * \rho_{j_n} - (\zeta_k)_t b_2^k\|_{L^1(Q)} \le \dfrac{\epsilon}{n+1}. \end{cases}$$

Defining

$$\begin{cases} F^\epsilon = \displaystyle\sum_{k=0}^n (\zeta_k b_0^k) * \rho_{j_n} \in (C_c^\infty(Q))^N, \\ G_1^\epsilon = \displaystyle\sum_{k=0}^n (\zeta_k B_1^k) * \rho_{j_n} \in (C_c^\infty(Q))^N, \\ g_2^\epsilon = -\displaystyle\sum_{k=0}^n (\zeta_k b_2^k) * \rho_{j_n} \in C_c^\infty(Q), \\ f^\epsilon = h_n + \displaystyle\sum_{k=0}^n (\zeta_k b_3^k) * \rho_{j_n} - \sum_{k=0}^n (f_0^k \nabla\zeta_k) * \rho_{j_n} \\ \qquad + \displaystyle\sum_{k=0}^n (B_1^k \nabla\zeta_k) * \rho_{j_n} + \sum_{k=0}^n ((\zeta_k)_t b_2^k) * \rho_{j_n} \in C_c^\infty(Q). \end{cases}$$

Then by (3.2.8) and (3.2.9), we get

$$\begin{cases} \|F^\epsilon - F\|_{(L^{p'(\cdot)}(Q))^N} \le 2\epsilon, \\ \|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \le 2\epsilon, \\ \|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \le 2\epsilon, \\ \|f^\epsilon - f\|_{L^1(Q)} \le 2\epsilon. \end{cases}$$

Let us write μ^ϵ as follows $\mu^\epsilon = f^\epsilon + F^\epsilon + \operatorname{div}(G_1^\epsilon) + (g_2^\epsilon)_t \in C_c^\infty(Q)$; it remains to prove that $\|\mu^\epsilon\|_{L^1(Q)} \le C$ with C not depending on ϵ. To see this, we recall that $((\zeta_k b_0^k) * \rho_{j_n}, (\zeta_k B_1^k) * \rho_{j_n}, (\zeta_k b_2^k) * \rho_{j_n}, (\zeta_k b_3^k) * \rho_{j_n}, (-f_0^k \nabla\zeta_k) * \rho_{j_n}, (-B_1^k \nabla\zeta_k) * \rho_{j_n}, ((\zeta_k)_t b_2^k) * \rho_{j_n})$ is a pseudo-decomposition of $(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_{j_n}$, we have

$$\begin{aligned} \mu^\epsilon &= h_n + \sum_{k=0}^n (\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_{j_n} \\ &= h_n + (\sum_{k=0}^n (\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas})) * \rho_{j_n} \\ &= h_n + g_n^{meas} * \rho_{j_n}. \end{aligned}$$

According to [**DPP**], $g_n^{meas} = m_n - h_n$. Then, it follows that $\|\mu^\epsilon\|_{L^1(Q)} \le 2\|h_n\|_{L^1(Q)} + \|m_n\|_{\mathcal{M}_b(Q)}$. Since h_n converges in $L^1(Q)$ and m_n converges in $\mathcal{M}_b(Q)$, $\|h_n\|_{L^1(Q)}$ and $\|m_n\|_{\mathcal{M}_b(Q)}$ are bounded. As consequence we have the desired majoration on $\|\mu^\epsilon\|_{L^1(Q)}$. □

3.3. Renormalized solutions and main result

As we said before, the notion of renormalized solutions was first introduced by DiPerna and Lions in [**DL1, DL2**] for the study of Boltzmann equation, it was then adapted to the study of some nonlinear elliptic and parabolic problems in fluid mechanics. Recently, this framework was extended to related problems with measures as data and variable exponent problems in [**OO**], where S. Ouaro and A. Ouédraogo studied a parabolic problem involving $p(x)-$Laplacian type operator and obtained the existence and uniqueness of entropy solutions for L^1-data, as well as integrability results for the solution and its gradient. The proofs rely crucially on the semigroup theory. Besides, Bendahmane and al. proved the existence and uniqueness of renormalized solutions of the same problem in [**BWZ**] using a priori estimates in Marcinkiewicz spaces with variable exponents, Zhang and Zhou [**ZZ**] uses a different method to prove the equivalence for the two notions. Inspired by these works, we define a notion of renormalized solutions for problem (3.2.4) with measure data. we are naturally led to introduce the functional space

$$
\begin{aligned}
X = \{u : \overline{\Omega} \times (0,T) \to \mathbb{R} \text{ is measurable such that } T_k(u) \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)),\\
\text{with } |\nabla T_k(u)| \in (L^{p(\cdot)}(Q))^N, \text{ for every } k > 0\},
\end{aligned} \tag{3.3.1}
$$

which, endowed with the norm (or, the equivalence norm)

$$\|u\|_X := \|\nabla u\|_{L^{p(\cdot)}(Q)}, \text{ or } \|u\|_X := \|u\|_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)},$$

X is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}(Q) \hookrightarrow L^{p_-}(0,T; L^{p(\cdot)}(\Omega))$ and the Poincaré inequality. and the truncation function at level k $T_k(s) = \max(-k, \min(k, s))$ and its primitive function $\Theta_k(z) = \int_0^z T_k(s)\, ds$. A function v such that $T_k(v) \in X$, for all $k > 0$, does not necessarily belongs to $L^1(0,T; W_0^{1,1}(\Omega))$. Thus ∇v in our equations is defined in a very weak sense.

Definition 3.20. For every measurable function $v : \overline{\Omega} \times (0,T) \to \mathbb{R}$ such that $T_k(v) \in X$ for all $k > 0$, there exists a unique measurable function $w : Q \to \mathbb{R}^N$, which we call the very weak gradient of v and denote $w = \nabla v$, such that

$$\nabla T_k(v) = w\chi_{\{|v|<k\}} \text{ a.e. in } \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if v belongs to $L^1(0,T; W_0^{1,1}(\Omega))$, then w coincides with the weak gradient of v.

Now, let us define $\mu_0 = \mu - g_2 = f + F - \operatorname{div}(G)$ where g_2 is the time-derivative part of μ. In view of the definition given in [**DPP**] and the preceding remarks, we have the following definition

Definition 3.21. Let $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$. We say that a measurable function u is a renormalized solution of the problem (3.2.4) if, for all k, $T > 0$, we have

$$u - g_2 \in L^\infty(0,T; L^1(\Omega)), \quad T_k(u) \in X, \tag{3.3.2}$$

$$\lim_{n\to\infty} \int_{\{n \le |u-g_2| \le n+1\}} |\nabla u|^{p(x)} dxdt = 0. \tag{3.3.3}$$

Moreover, for all $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$
\begin{aligned}
&- \int_Q S(u_0)\varphi(0)dx - \int_0^T \langle \varphi_t, S(u-g_2)\rangle dt + \int_Q S'(u-g_2)a(t,x,\nabla u)\cdot \nabla\varphi \, dxdt\\
&+ \int_Q S''(u-g_2)a(t,x,\nabla u)\cdot \nabla(u-g_2)\varphi \, dxdt = \int_Q S'(u-g_2)\varphi d\mu_0,
\end{aligned} \tag{3.3.4}
$$

for every $\varphi \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$ with $\nabla\varphi \in (L^{p(\cdot)}(Q))^N$, $\varphi_t \in L^{p'_-}(0,T; W^{-1,p'(\cdot)}(\Omega))$ with $\varphi(T) = 0$ such that $S'(u-g_2)\varphi \in X$, and

$$S(u-g_2)(0) = S(u_0) \text{ in } L^1(\Omega). \tag{3.3.5}$$

Remark 3.22. First of all, notice that, thanks to our regularity assumptions and the choice of S', all terms in (3.3.4) are well defined, also observe that (3.3.4) implies that equation

$$
\begin{aligned}
&(S(u-g_2))_t - \operatorname{div}(a(t,x,\nabla u)S'(u-g_2)) + S''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2)\\
&= S'(u-g_2)f + S''(u-g_2)F\cdot\nabla(u-g_2) - \operatorname{div}(FS'(u-g_2))\\
&+ S''(u-g_2)G\cdot\nabla(u-g_2) - \operatorname{div}(GS'(u-g_2))
\end{aligned} \tag{3.3.6}
$$

is satisfied in the sense of distributions since $T_k(u-g_2)$ belongs to X for every $k > 0$ and since S' has compact support. Indeed by taking M such that Supp $S' \subset (-M, M)$, since $S'(u-g_2) = S''(u-g_2) = 0$ as soon as $|u-g_2| \ge M$, we can replace, everywhere in (3.3.4), $\nabla(u-g_2)$ by $\nabla T_M(u-g_2) \in (L^{p(\cdot)}(Q))^N$ and ∇u

by $\nabla(T_M(u-g_2)) + \nabla g_2 \in (L^{p(\cdot)}(Q))^N$. Moreover, according to the assumption (3.2.2) and the definition of ∇u, $\nabla u = \nabla(u-g_2) + \nabla g_2$, we have $\nabla(u-g_2)$ is well defined and $|a(t,x,\nabla u)| \in L^{p'(x)}(Q)$. We also have, for all S as above, $S(u-g_2) = S(T_M(u-g_2)) \in X$ and $S'(u-g_2)f \in L^1(Q)$, $S'(u-g_2)F \in L^{p'(\cdot)}(Q)$, $S'(u-g_2)G_1 \in L^{p'(\cdot)}(Q)$, $S'(u-g_2)a(t,x,\nabla u) \in (L^{p'(\cdot)}(Q))^N$, $S''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2) \in L^1(Q)$, $S''(u-g_2)F\cdot\nabla(u-g_2) \in L^1(Q)$ and $S''(u-g_2)G_1\cdot\nabla(u-g_2) \in L^1(Q)$. Thus, by equation (3.3.6), $(S(u-g_2))_t$ belongs to the space $X' + L^1(Q)$, and therefore $S(u-g_2)$ belongs to $C([0,T];L^1(\Omega))$, one can say that the initial datum is achieved in a weak sense, that is $S(u-g_2)(0) = S(u_0)$ in $L^1(\Omega)$ for every renormalization S. Note also that, since $S(u-g_2)_t \in X' + L^1(Q)$, we can use in (3.3.4) not only functions in $C_0^\infty(Q)$ but also in $X \cap L^\infty(Q)$.

Remark 3.23. Observe that (3.3.3) implies

$$\lim_{n\to\infty}\int_{\{n\le|u-g_2|\le n+c\}} |\nabla(u-g_2)|^{p(x)}dxdt = 0, \quad \text{for all } c>0. \tag{3.3.7}$$

Remark 3.24. Let us denote by $v = u - g_2$ the solution of (3.2.4), since $S(v) \in X\cap L^\infty(Q)$ and $(S_n(v))_t \in X^\star + L^1(Q)$ and thanks to Theorem 3.8, $S_n(v)$ turns out to admit a $\mathrm{cap}_{p(\cdot)}$ quasi-continuous representative finite $\mathrm{cap}_{p(\cdot)}$ quasi-everywhere.

For classical Sobolev spaces, the definition of renormalized solution does not depend on the decomposition of the measures μ as shown in Proposition 3.10 in **[DPP]**. Next result try to stress the fact that even for generalized Sobolev spaces this fact should be true.

Proposition 3.25. *Let u be a renormalized solution of* (3.2.4)*. Then u satisfies Definition 3.21 for every decomposition $(\tilde f, \tilde F, -\operatorname{div}(\tilde G_1), \tilde g_2)$ such that $g_2 - \tilde g_2 \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$.*

Proof. Assume that u satisfies Definition 3.21 for $(f, F, -\operatorname{div}(G), g_2)$ and let $(\tilde f, \tilde F, -\operatorname{div}(\tilde G), \tilde g_2)$ be a different decomposition of μ_0 such that $g_2 - \tilde g_2$ is bounded. Thanks to Lemma 3.15, we readily have that $\tilde v = \tilde u - \tilde g_2 \in L^\infty(0,T;L^1(\Omega))$. To prove that $T_k(u-\tilde g_2) \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $\nabla T_k(u-\tilde g_2) \in L^{p(\cdot)}(Q)$ with $k>0$ we can reason as in **[DPP]** with $S = S_n$ and we choose as test function $T_k(S_n(u-g_2)+g_2-\tilde g_2) \in X\cap L^\infty(Q)$ in (3.3.4). Thanks to Lemma 3.15 we have

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g_2)+g_2-\tilde g_2)_t, T_k(S_n(u-g_2)+g_2-\tilde g_2)\rangle dt && (A)\\
&+\int_Q S_n'(u-g_2)a(t,x,\nabla u)\cdot\nabla T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (B)\\
&= -\int_Q S_n''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (C)\\
&+\int_Q ((S_n'(u-g_2)-1)f+\tilde f)T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (D)\\
&+\int_Q (S_n'(u-g_2)-1)F+\tilde F)\cdot\nabla T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (E)\\
&+\int_Q ((S_n'(u-g_2)-1)G_1+\tilde G_1)\cdot\nabla T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (F)\\
&+\int_Q S_n''(u-g_2)F\cdot\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (G)\\
&+\int_Q S_n''(u-g_2)G\cdot\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde g_2)dxdt && (H)
\end{aligned} \tag{3.3.8}$$

Let us analyze term by term the above identity. First of all, concerning the first term of (3.3.8) we integrate in time to get

$$\begin{aligned}(A) &= \int_0^T \langle (S_n(u-g_2)+g_2-\tilde{g}_2)_t, T_k(S_n(u-g_2)+g_2-\tilde{g}_2)\rangle dt = \left[\int_\Omega \Theta_k(u-g_2)+g_2-\tilde{g}_2)dx\right]_0^T \\ &= \int_\Omega \Theta_k(S_n(u-g_2))(T)+(g_2-\tilde{g}_2)(T)dx - \int_\Omega \Theta_k(S_n(u-g_2))(0)+(g_2-\tilde{g}_2)(0)dx.\end{aligned}$$

Since $S_n(u-g_2)(0)=S_n(u_0)$ and $(g_2-\tilde{g}_2)(0)=0$, we have $S_n(u-g_2)(0)+(g_2-\tilde{g}_2)(0)=S_n(u_0)$, so that using $0\leq\Theta_k(s)\leq k(s)$, the first term of (3.3.8),

$$(A)\leq k\|u_0\|_{L^1(\Omega)}$$

On the other hand, since $|S_n''(s)|\leq 1$ and $S_n''(s)\neq 0$ if $|s|\in[n,n+1]$, using (3.2.2) and Young's inequality

$$\begin{aligned}|(C)+(G)+(H)| &\leq \beta k\|S_n'(s)\|_{L^\infty(\mathbb{R})}\int_{\{n\leq|u-g_2|\leq n+1\}} |(b(t,x)+|\nabla u|^{p(x)-1})||\nabla(u-g_2)| \\ &\leq Ck[\int_{\{n\leq|u-g_2|\leq n+1\}} \frac{p^+-1}{p_-}(|b(t,x)|^{p'(x)}+|G_1|^{p'(x)}+|\nabla u|^{p'(x)(p(x)-1)})dxdt \\ &\quad+\int_{\{n\leq|u-g_2|\leq n+1\}}(|\nabla u|^{p(x)}+|\nabla g_2|^{p(x)})dxdt] \\ &\leq Ck[\int_{\{n\leq|u-g_2|\leq n+1\}}(|b(t,x)|^{p'(x)}+|F|^{p'(x)}+|G_1|^{p'(x)}+|\nabla g_2|^{p'(x)})dxdt \\ &\quad+\int_{\{n\leq|u-g_2|\leq n+1\}}|\nabla u|^{p(x)}dxdt].\end{aligned}$$

By the fact that $\text{meas}\{n\leq|u-g_2|\leq n+1\}\underset{n\to\infty}{\to}0$ and using (3.3.3), we get

$$|(C)+(G)+(H)|\leq\omega(n),$$

where $\omega(n)$ tends to zero as $n\to\infty$. Now, if $E_n=\{|S_n(u-g_2)+g_2-\tilde{g}_2|\leq k\}$ we have (recalling that if $0\leq S_n'(s)\leq 1$ then $|S_n'(s)|^{p'(x)}\leq S_n'(s)$),

$$\begin{aligned}|(D)+(E)+(F)| &\leq \int_Q(|f|+|\tilde{f}|)|T_k(S_n(u-g_2)+g_2-\tilde{g}_2)|dxdt \\ &\quad+\int_{E_n}(|F|+|\tilde{F}|)(S_n'(u-g_2)|\nabla(u-g_2)|+|\nabla g_2|+|\nabla\tilde{g}_2|)dxdt \\ &\quad+\int_{E_n}(|G_1|+|\tilde{G}_1|)(S_n'(u-g_2)|\nabla(u-g_2)|+|\nabla g_2|+|\nabla\tilde{g}_2|)dxdt \\ &\leq k(\|f\|_{L^1(Q)}+\|\tilde{f}\|_{L^1(Q)})+\int_{E_n}(|F_1|+|\tilde{F}_1|)S_n'(u-g_2)|\nabla u|dxdt \\ &\quad+2\int_Q(|F_1|+|\tilde{F}_1|)(|\nabla g_2|+|\nabla\tilde{g}_2|)dxdt+\int_{E_n}(|G_1|+|\tilde{G}_1|)S_n'(u-g_2)|\nabla u|dxdt \\ &\quad+2\int_Q(|G_1|+|\tilde{G}_1|)(|\nabla g_2|+|\nabla\tilde{g}_2|)dxdt \\ &\leq k(\|f\|_{L^1(Q)}+\|\tilde{f}\|_{L^1(Q)}) \\ &\quad+2\frac{p^+-1}{p_-}\int_Q|F|^{p'(x)}+|\tilde{F}|^{p'(x)}+|G_1|^{p'(x)}+|\tilde{G}_1|^{p'(x)}dxdt \\ &\quad+\frac{2}{p_-}\int_{\{n\leq|u-g_2|\leq n+1\}}|\nabla u|^{p(x)}dxdt+\frac{2}{p_-}\int_Q|\nabla g_2|^{p(x)}+|\nabla\tilde{g}_2|^{p(x)}dxdt \\ &\leq C+\omega(n).\end{aligned}$$

□

Our main result is the following Theorem

THEOREM 3.26. *Let $1 < p_- \le p_+ < N$, and suppose that $p_- > \frac{2N+1}{N+1}$. Assume that* (3.2.1) − (3.2.3) *hold true, $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$. Then there exists a renormalized solution u of problem* (3.2.4).

3.4. Proof of the main result

We can now start the proof of Theorem 3.26. Following a standard approach, we obtain the existence of a solution as limit of regular problems. For this purpose we consider the approximate problem

$$\begin{cases} u_t^\epsilon - \operatorname{div}(a(t,x,\nabla u^\epsilon)) = \mu^\epsilon & \text{in } (0,T)\times\Omega, \\ u^\epsilon(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u^\epsilon(0,x) = u_0^\epsilon(x) & \text{in } \Omega, \end{cases} \tag{3.4.1}$$

where $\{\mu^\epsilon\}_{\epsilon>0}$, $\{u_0^\epsilon\}_{\epsilon>0}$ are smooth approximations of the data μ and u_0 with

$$\|u_0^\epsilon\|_{L^1(\Omega)} \le C\|u_0\|_{L^1(\Omega)},\ \|\mu^\epsilon\|_{L^1(Q)} \le C|\mu|.$$

Hence by the standard theory of monotone operators **[LL]** or using Lemma 2.5 of **[ZZ]** with rather minor modifications, there exists a variational solution u^ϵ for each $\epsilon > 0$. Moreover, from Theorem 3.13, there exists a decomposition $(f^\epsilon, F^\epsilon, \operatorname{div}(G_1^\epsilon), g_2^\epsilon)$ of μ^ϵ with $f^\epsilon \in C_c^\infty(Q)$ such that $\|f^\epsilon - f\|_{L^1(Q)} \le C\epsilon$, $F^\epsilon \in (C_c^\infty(Q))^N$ such that $\|F^\epsilon - F\|_{(L^{p(\cdot)}(Q))^N} \le C\epsilon$, $G_1^\epsilon \in (C_c^\infty(Q))^N$ such that $\|G_1^\epsilon - G_1\|_{(L^{p(\cdot)}(Q))^N} \le C\epsilon$ and $g_2^\epsilon \in C_c^\infty(Q)$ such that $\|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \le C\epsilon$ (with C not depending on ϵ) such that

$$\begin{aligned} &\int_0^t \langle (u^\epsilon - g_2^\epsilon)_t, \varphi\rangle ds + \int_0^t\int_\Omega a(s,x,\nabla u^\epsilon)\cdot\nabla\varphi\, dxds \\ &= \int_0^t\int_\Omega f^\epsilon \varphi\, dxds + \int_0^t\int_\Omega F\cdot\nabla\varphi\, dxds + \int_0^t \langle \operatorname{div}(G_1^\epsilon), \varphi\rangle ds, \end{aligned} \tag{3.4.2}$$

$\forall \varphi \in L^{p_-}(0,T;V)$ with $\nabla\varphi \in (L^{p(\cdot)}(Q))^N$, $\forall t \in [0,T]$. Next, following the ideas of **[BDGO]** (see also **[DO1]**), we can perform some estimates for the sequence $(u^\epsilon)_{\epsilon>0}$, to prove that u is actually the renormalized solution to the parabolic problem (3.2.4).

PROPOSITION 3.27. *Let u^ϵ as defined before, then*

$$\begin{cases} \|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \\ \displaystyle\int_Q |\nabla T_k(u^\epsilon)|^{p(x)} dxdt \le Ck, \\ \|u^\epsilon - g_2^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \\ \displaystyle\int_Q |\nabla T_k(u^\epsilon - g_2^\epsilon)|^{p(x)} dxdt \le C(k+1). \end{cases} \tag{3.4.3}$$

Moreover, there exists a measurable functions u and $v = u - g_2$ such that $T_k(u)$ and $T_k(v)$ belongs to X, u and v belongs to $L^\infty(0,T;L^1(\Omega))$; and, up to a subsequence, for any $k > 0$, and for every $q(\cdot) < p(\cdot) - \frac{N}{N+1}$, we have

$$\begin{cases} u^\epsilon \to u \text{ a.e. in } Q \text{ weakly in } L^{q_-}(0,T;W_0^{1,q(\cdot)}(\Omega)) \text{ and strongly in } L^1(Q), \\ (u^\epsilon - g_2^\epsilon) \to (u - g_2) \text{ a.e. in } Q \text{ weakly in } L^{q_-}(0,T;W_0^{1,q(\cdot)}(\Omega)) \text{ and strongly in } L^1(Q), \\ (T_k(u^\epsilon) \rightharpoonup T_k(u) \text{ weakly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \text{ and a.e. on } Q, \\ T_k(u^\epsilon - g_2^\epsilon) \rightharpoonup T_k(u - g_2) \text{ weakly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \text{ and a.e. on } Q, \\ \nabla u^\epsilon \to \nabla u \text{ a.e. in } Q, \\ \nabla(u^\epsilon - g_2^\epsilon) \to \nabla(u - g_2) \text{ a.e. in } Q, \end{cases}$$

PROOF. Here we give an idea on how (3.4.3) can be obtained following the outlines of **[DPP]**. Let $\epsilon > 0$, by taking $T_k(u^\epsilon)$ as test function in (3.4.1), we obtain

$$\int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon)\rangle dt + \int_Q a(t,x,\nabla u^\epsilon)\cdot\nabla T_k(u^\epsilon) dxdt = \int_Q \mu^\epsilon T_k(u^\epsilon) dxdt.$$

We have $\Theta_k(r) = \int_0^r T_k(s)ds$ and $|\Theta_k(r)| \leq k|r|$, then

$$\int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon)\rangle dt = \int_\Omega \int_0^t \frac{\partial u^\epsilon}{\partial t} T_k(u^\epsilon) dxdt = \int_\Omega \int_0^t \frac{\partial \Theta_k(u^\epsilon)}{\partial t} dxdt$$
$$= \int_\Omega \Theta_k(u^\epsilon(T))dx - \int_\Omega \Theta_k(u_0^\epsilon)dx \geq \int_\Omega \Theta_k(u^\epsilon(t))dx - k\|u_0^\epsilon\|_{L^1(\Omega)}.$$

From (3.2.1) and using the fact that $\|u_0^\epsilon\|_{L^1(\Omega)}$ and $\|\mu^\epsilon\|_{L^1(Q)}$ are bounded, then

$$\int_\Omega \Theta_k(u^\epsilon(t))dx + \int_0^t \int_\Omega |\nabla T_k(u^\epsilon)|^{p(x)} dxdt \leq Ck, \quad \forall k \geq 0, \ \forall t \in [0,T].$$

Since $\Theta_k(s)$ is nonnegative and $|\Theta_1(s)| \geq |s| - 1$ for $k = 1$, we get

$$\int_\Omega |u^\epsilon(t)|dx + \int_0^t \int_\Omega |\nabla T_k(u^\epsilon)|^{p(x)} dxdt \leq C(k+1) \quad \forall t \in [0,T]. \tag{3.4.4}$$

Taking the supremum on $(0,T)$, we obtain the estimate

$$\|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C.$$

To prove the estimate of $u^\epsilon - g_2^\epsilon$ in $L^\infty(0,T;L^1(\Omega))$, we will use the test function $T_k(u^\epsilon - g_2^\epsilon)$ in (3.4.2), this gives

$$\int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon - g_2^\epsilon)\rangle dxdt - \int_0^t \langle (g_2^\epsilon)_t, T_k(u^\epsilon - g_2^\epsilon)\rangle dt + \int_Q a(t,x,\nabla u^\epsilon) \cdot \nabla T_k(u^\epsilon - g_2^\epsilon)dxdt$$
$$= \int_Q f^\epsilon T_k(u^\epsilon - g_2^\epsilon)dxdt + \int_Q F \cdot \nabla T_k(u^\epsilon - g_2^\epsilon)dxdt - \int_0^t \langle \mathrm{div}(G_1^\epsilon), T_k(u^\epsilon - g_2^\epsilon)\rangle.$$

Now, since g_2^ϵ has compact support in Q, so that $(u^\epsilon - g_2^\epsilon)(0) = u^\epsilon(0) = u_0^\epsilon$. Using the integration by parts in time in the first term and using (3.2.1) we get

$$\int_\Omega \Theta_k(u^\epsilon - g_2^\epsilon)(t)dx - \int_\Omega \Theta_k(u_0^\epsilon)dx + \alpha \int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} |\nabla u^\epsilon|^{p(x)} dxdt - \int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} a(t,x,\nabla u^\epsilon) \cdot \nabla g_2^\epsilon \, dxdt$$
$$\leq \int_Q f^\epsilon T_k(u^\epsilon - g_2^\epsilon)dxdt + \int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} F \cdot \nabla(u^\epsilon - g_2^\epsilon)dxdt + \int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} G_1^\epsilon \cdot \nabla(u^\epsilon - g_2^\epsilon)dxdt.$$

Young's inequality then implies, using also (3.2.2),

$$\int_\Omega \Theta_k(u^\epsilon - g_2^\epsilon)(t)dx + \alpha \int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} |\nabla u^\epsilon|^{p(x)} dxdt$$
$$\leq C\beta \left[\int_Q |b(t,x)|^{p'(x)} dxdt + \int_Q |\nabla u^\epsilon|^{p(x)} dxdt + \int_Q |\nabla g_2^\epsilon|^{p(x)} dxdt \right]$$
$$+ k\left[\|u_0^\epsilon\|_{L^1(\Omega)} + \|f^\epsilon\|_{L^1(Q)}\right] + \frac{\alpha}{2}\left[\int_Q |\nabla u|^{p(x)} dxdt + \int_Q |\nabla g_2^\epsilon|^{p(x)} dxdt\right]$$
$$+ C_\alpha \left[\int_Q |F|^{p'(x)} dxdt + \int_Q |G_1^\epsilon|^{p'(x)} dxdt\right],$$

where C_α denote a positive constant which depends on p_+ and p_- but not depending on ϵ and k. In the same way we can deal with the right hand side of the last inequality, we used the fact that $f^\epsilon \in L^1(Q)$, $F^\epsilon \in (L^{p'(\cdot)}(Q))^N$, $g_1^\epsilon \in L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0,T;V)$ and $u_0^\epsilon \in L^1(\Omega)$, (note that $\Theta_k(s)$ is nonnegative for any $k \geq 0$) to obtain

$$\begin{cases} \Theta_1(u^\epsilon - g_2^\epsilon)(t) \leq C, \quad \forall t \in [0,T] \\ \displaystyle\int_{\{|u^\epsilon - g_2^\epsilon| \leq k\}} |\nabla u^\epsilon|^{p(x)} dxdt \leq C(k+1). \end{cases}$$

Moreover, using the boundedness of g_2^ϵ in V, we have

$$\begin{cases} \|u^\epsilon - g_2^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ \displaystyle\int_Q |\nabla T_k(u^\epsilon - g_2^\epsilon|^{p(x)} dxdt \leq C(k+1). \end{cases}$$

Now, we shall use the above estimates to prove some compactness results that will be useful to pass to the limit in the renormalized formulation for u^ϵ.

If we multiply the first equation in (3.4.1) by $\gamma_k'(u^\epsilon - g_2^\epsilon)$ where γ is a $C^2(\mathbb{R})$ nondecreasing function such that $\gamma(s) = s$ for $|s| \leq \frac{k}{2}$ and $\gamma(s) = k$ for $|s| > k$, remark that γ_k' and γ_k'' has compact support, we get

$$\begin{aligned}(\gamma_k(u^\epsilon - g_2^\epsilon))_t &- \operatorname{div}(a(t,x,\nabla u^\epsilon)\gamma_k'(u^\epsilon - g_2^\epsilon)) + \gamma_k''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon)\\ &= \gamma_k'(u^\epsilon - g_2^\epsilon)f^\epsilon - \operatorname{div}(F^\epsilon\gamma_k'(u^\epsilon - g_2^\epsilon)) + \gamma_n''(u^\epsilon - g_2^\epsilon)F^\epsilon\cdot\nabla(u^\epsilon - g_2^\epsilon)\\ &+ \gamma_k''(u^\epsilon - g_2^\epsilon)G_1\cdot\nabla(u^\epsilon - g_2^\epsilon) - \operatorname{div}(G_1^\epsilon\gamma_k'(u^\epsilon - g_2^\epsilon)).\end{aligned}\tag{3.4.5}$$

We also have $\gamma_k''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k''(u^\epsilon - g_2^\epsilon)F^\epsilon\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k''(u^\epsilon - g_2^\epsilon)G_1\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon) \in (L^{p'(\cdot)}(Q))^N$, $\gamma_k'(u^\epsilon - g_2^\epsilon)G_1^\epsilon \in (L^{p'(\cdot)}(Q))^N$, $\gamma_k'(u^\epsilon - g_2^\epsilon)F^\epsilon \in (L^{p'(\cdot)}(Q))^N$. Thus, by equation (3.4.5), $(\gamma_k(u^\epsilon - g_2^\epsilon))_t$ belong to the space $X^* + L^1(Q)$. On the other hand, by the last equality $T_k(u^\epsilon - g_2^\epsilon)$ is bounded in X for any $k > 0$, then we have

$$\begin{aligned}k\ \operatorname{meas}\{|u^\epsilon - g_2^\epsilon| > k\} &= \int_{\{|u^\epsilon - g_2^\epsilon| > k\}} |T_k(u^\epsilon - g_2^\epsilon)|dxdt \leq \int_Q |T_k(u^\epsilon - g_2^\epsilon)|dxdt\\ &\leq 2(\ \operatorname{meas}(Q)+1)^{\frac{1}{p'_-}}\|T_k(u^\epsilon - g_2^\epsilon)\|_X \leq Ck^{\frac{1}{p_-}},\end{aligned}$$

which implies that

$$\operatorname{meas}\{|u^\epsilon - g_2^\epsilon| > k\} \leq C\frac{1}{k^{1-\frac{1}{p_-}}} \to 0 \text{ as } k \to \infty.\tag{3.4.6}$$

Let $n, m \geq 0$, for all $\lambda > 0$, we have

$$\begin{aligned}\operatorname{meas}\{|(u^n - g_2^n)| > \lambda\} &\leq \operatorname{meas}\{|u_n - g_2^n| > k\} + \operatorname{meas}\{|u_m - g_2^m)| > k\}\\ &+ \operatorname{meas}\{|T_k(u_n - g_2^n) - T_k(u_m - g_2^m)| > \lambda\}.\end{aligned}\tag{3.4.7}$$

Using (3.4.6) we get that for all $\epsilon > 0$, there exists $k_0 > 0$ such that $\forall k \geq k_0(\epsilon)$,

$$\operatorname{meas}\{|u_n - g_2^n| > k\} \leq \frac{\epsilon}{3}, \quad \operatorname{meas}\{|u_m - g_2^m| > k\} \leq \frac{\epsilon}{3}.$$

On the other hand, we have $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ is bounded in X. Then, there exists a sequence still denoted $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ such that

$$T_k(u_n - g_2^n) \rightharpoonup \eta_k \text{ in } X \text{ as } n \to \infty$$

and by the compact embedding $\{u \in X \text{ such that } u_t \in X^*\}$ in $L^1(Q)$, we obtain

$$T_k(u_n - g_2^n) \to \eta_k \text{ in } L^1(Q) \text{ and a.e. in } Q.$$

Thus, we can assume that $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Q, therefore for all $k > 0$ and $\lambda, \epsilon > 0$ there exists $n_0 = n_0(k, \lambda, \epsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n - g_2^n) - T_k(u_m - g_2^m)| > \lambda\} \leq \frac{\epsilon}{3} \quad \forall n, m \geq n_0.\tag{3.4.8}$$

By combining $(3.4.6) - (3.4.8)$, we deduce that for all $\epsilon, \lambda > 0$ there exits $n_0 = n_0(\lambda, \epsilon)$ such that

$$\operatorname{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \lambda\} \leq \epsilon \quad \forall n, m \geq n_0.$$

It follows that $(u^\epsilon - g_2^\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u^\epsilon - g_2^\epsilon)_{\epsilon>0}$ such that

$$\begin{cases} u^\epsilon - g_2^\epsilon \to u - g_2 & \text{a.e. in } Q,\\ T_k(u^\epsilon - g_2^\epsilon > 0) \rightharpoonup T_k(u - g_2) & \text{weakly in } X.\end{cases}$$

In the view of the strong convergence of g_2^ϵ to g_2 in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, we have

$$\begin{cases} u^\epsilon \to u & \text{a.e. in } Q,\\ T_k(u^\epsilon) \rightharpoonup T_k(u) & \text{weakly in } X.\end{cases}$$

Finally, the sequence $u^\epsilon - g_2^\epsilon$ satisfies the hypotheses of [**BDGO**], and so we get

$$\begin{cases} \nabla(u^\epsilon - g_2^\epsilon) \to \nabla(u - g_2) & \text{a.e. in } Q, \\ \nabla u^\epsilon \to \nabla(u) & \text{a.e. in } Q. \end{cases}$$

Next we shall prove the strong convergence of truncates of renormalized solutions of problem (3.2.4). To do that we will crossover the approach used in [**Po1**]. With the symbol $T_k(v)_\mu$ we indicate the Landes time-regularization of the truncate function $T_k(v)$; this notion, introduced in [**La**], was fruitfully used in several papers afterwards (see in particular [**BDGO, BP, DO1**]). Let z_μ be a sequence of functions such that

$$\begin{cases} z_\mu \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \ \|z_\mu\|_{L^\infty(\Omega)} \leq k, \\ z_\mu \to T_k(u_0) \text{ a.e. in } \Omega \text{ as } \mu \text{ tends to infinity}, \\ \dfrac{1}{\mu}\|z_\mu\|_{W_0^{1,p(\cdot)}(\Omega)} \to 0 \text{ as } \mu \text{ tends to infinity}. \end{cases}$$

Then, for fixed $k > 0$, and $\mu > 0$, we denote by $T_k(v)_\mu$ the unique solution of the problem

$$\begin{cases} (T_k(v)_\mu)_t = \mu(T_k(v) - T_k(v)_\mu) \text{ in the sense of distributions}, \\ T_k(v)_\mu(0) = z_\mu \text{ in } \Omega. \end{cases}$$

Therefore $T_k(v)_\mu \in X \cap L^\infty(Q)$ and $\frac{d}{dt}T_k(v) \in V$, and it can be proved, see [**La**], that up to subsequences

$$\begin{cases} T_k(v)_\mu \to T_k(v) \text{ strongly in } X \text{ and a.e. in } Q, \\ \|T_k(v)_\mu\|_{L^\infty(Q)} \leq k, \quad \forall \mu > 0. \end{cases}$$

Choosing w^ϵ as a test function in the formulation (3.4.2), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega (v^\epsilon)_t w^\epsilon \, dxdt + \int_0^T \int_\Omega a(t,x,\nabla u^\epsilon) \cdot \nabla w^\epsilon \, dxdt \\ &= \int_0^T \int_\Omega f^\epsilon w^\epsilon \, dxdt + \int_0^T \int_\Omega F^\epsilon \cdot \nabla w^\epsilon \, dxdt + \int_0^T \langle g_1^\epsilon, w^\epsilon \rangle \, dxdt. \end{aligned} \tag{3.4.9}$$

So, for the first term on the right-hand side of (3.4.9), we have

$$\begin{aligned} \left| \int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt \right| &\leq \int_0^T \int_\Omega |f^\epsilon - f||T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt \\ &\quad + \int_0^T \int_\Omega |f T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt \\ &\leq 2k \int_0^T \int_\Omega |f^\epsilon - f|dxdt \\ &\quad + \int_0^T \int_\Omega |f T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt. \end{aligned}$$

By using the fact that f^ϵ is strongly compact in $L^1(Q)$, the weak convergence of $T_k(v^\epsilon)$ to $T_k(v)$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and a.e. in Q, the definition of $(T_k(v)_\mu)$ and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{h\to+\infty} \lim_{\mu\to+\infty} \lim_{\epsilon\to0} \left| \int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt \right| \leq \lim_{h\to+\infty} \int_0^T \int_\Omega |f T_{2k}(v - T_h(v))|dxdt = 0.$$

Using the notations $\omega(\epsilon,\mu,h)$, we obtain

$$\int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt = \omega(\epsilon,\mu,h), \ \int_0^T \int_\Omega F^\epsilon \cdot \nabla w^\epsilon dxdt = \omega(\epsilon,\mu,h). \tag{3.4.10}$$

Let us analyze the second term in (3.4.9). Due to the fact that $\nabla w^\epsilon = 0$ if $|v^\epsilon| > M = h + 4k$, observing that

$$\int_0^T \int_\Omega a(t,x,\nabla u^\epsilon) \cdot \nabla w^\epsilon dxdt = \int_0^T \int_\Omega a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon| \leq M\}}) \cdot \nabla w^\epsilon dxdt.$$

Next we split the integral in the sets $\{|v^\epsilon| \leq k\}$ and $\{|v^\epsilon| > k\}$, so that we have, recalling that for $h > 2k$,

$$
\begin{aligned}
&\int_0^T \int_\Omega a(t,x,\nabla u^\epsilon \chi_{\{|u^\epsilon|k\}}) \cdot \nabla T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu) dxdt \\
&= \int\int_{\{|v^\epsilon|\leq k\}} a(t,x,\nabla u^\epsilon) \cdot \nabla(v^\epsilon - T_k(v)_\mu) dxdt \\
&+ \int\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}}) \cdot \nabla(v^\epsilon - T_h(v^\epsilon)) dxdt \\
&- \int\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}}) \cdot \nabla T_k(v)_\mu dxdt \\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{3.4.11}
$$

Let us estimate I_2. Since $v^\epsilon = T_h(v^\epsilon) = 0$ if $|v^\epsilon| \leq h$, using (3.2.2) and young's inequality, we have

$$
\begin{aligned}
|I_2| &= |\int\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}}) \cdot \nabla(v^\epsilon - T_h(v^\epsilon)) dxdt| \\
&\leq \int\int_{\{h\leq|v^\epsilon|\leq M\}} |a(t,x,\nabla u^\epsilon)||\nabla v^\epsilon| dxdt \\
&\leq \int\int_{\{h\leq|v^\epsilon|\leq M\}} \beta(b(t,x) + |\nabla u^\epsilon|^{p(x)-1})|\nabla(u^\epsilon - g_2^\epsilon)| dxdt \\
&\leq \int\int_{\{h\leq|v^\epsilon|\leq M\}} \beta(b(t,x)|\nabla(u^\epsilon - g_2^\epsilon)|) dxdt + \int\int_{\{h\leq|v^\epsilon|\leq M\}} C|\nabla u|^{p(x)-1}|\nabla(u^\epsilon - g_2^\epsilon)| dxdt \\
&\leq \int\int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|b(t,x)|^{p'(x)} dxdt + \int\int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla u^\epsilon|^{p(x)} dxdt \\
&+ \int\int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla g_2^\epsilon|^{p(x)} dxdt + \int\int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla u^\epsilon|^{p(x)} dxdt \\
&+ \int\int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla u^\epsilon|^{p(x)} dxdt + \int\int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p_-}|\nabla g_2^\epsilon|^{p(x)} dxdt \\
&\leq C\int\int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla u^\epsilon|^{p(x)} dxdt + C\int\int_{\{h\leq|v^\epsilon|\leq M\}} |b(t,x)|^{p'(x)} dxdt \\
&+ C\int\int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla g_2^\epsilon|^{p(x)} dxdt.
\end{aligned}
$$

Moreover, since $b(t,x)$ and $(\nabla u^\epsilon)_{\epsilon\geq 0}$ are bounded in $L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ respectively, and as meas$\{h \leq |v^\epsilon| < M\}$ converges uniformly to zero as h tends to infinity with respect to ϵ, then, thanks to the equi-integrability of $|\nabla g_2^\epsilon|^{p(x)}$, we can pass to the limit in (I_2) as $\epsilon \to 0$ and $h \to +\infty$ respectively, and using Lebesgue dominated convergence theorem, we easily get

$$I_2 = \omega(\epsilon, h).$$

It remains to estimate I_3, let us remark that, since $(\nabla u^\epsilon \chi_{|v^\epsilon|\leq M})$ is bounded in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, (3.2.2) implies that $(a(t,x,\nabla u^\epsilon)\chi_{\{|v^\epsilon|\leq M\}})_{\epsilon>0}$ is bounded in $L^{p'(\cdot)}(Q)$. The almost everywhere convergence of v^ϵ to v as $\epsilon \to 0$, implies that $|\nabla T_k(v)|\chi_{\{|v^\epsilon|\leq k\}}$ strongly converges to zero in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. So that by the Lebesgue dominated convergence theorem, we have

$$\limsup_{\epsilon\to 0} \int\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}}) \cdot \nabla T_k(v) dxdt = 0$$

and we readily have that

$$\begin{aligned}I_3 &= \int\int_{\{|v^\epsilon\}|>k} a(t, x\nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \cdot \nabla T_k(v)_\mu dxdt \\ &= \int\int_{\{|v^\epsilon|>k\}} a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \cdot \nabla (T_k(v)_\mu - T_k(v)) dxdt \\ &= \omega(\epsilon) + \int\int_{\{|v^\epsilon|>k\}} a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \cdot \nabla (T_k(v)_\mu - T_k(v)) dxdt.\end{aligned}$$

Observing that $(a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le M}))_{\epsilon>0}$ is bounded in $L^{p'(\cdot)}(Q)$ and thanks to the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in X, we can apply the Lebesgue Dominated Convergence theorem to obtain

$$\int\int_{\{|v^\epsilon|>k\}} a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \cdot \nabla (T_k(v)_\mu - T_k(v)) dxdt = \omega(\epsilon, \mu).$$

We can conclude that

$$I_3 = \omega(\epsilon, \mu).$$

On the other hand, using (3.4.11), according to the fact that I_2 and I_3 converge to zero, then

$$\int_0^T \int_\Omega a(t, x, \nabla u^\epsilon) \cdot \nabla w^\epsilon dxdt = \int\int_{\{|v^\epsilon| \le k\}} a(t, x, \nabla u^\epsilon) \cdot \nabla (v^\epsilon - T_k(v)_\mu) dxdt + \omega(\epsilon, \mu, h).$$

Moreover, (3.4.10) and (3.4.11) together with (3.4.9) yield

$$\int_0^T \int_\Omega (v^\epsilon)_t w^\epsilon dxdt + \int\int_{\{|v^\epsilon| \le k\}} a(t, x, \nabla u^\epsilon) \cdot \nabla (v^\epsilon - (T_k(v))_\mu) dxdt = \omega(\epsilon, \mu, h). \tag{3.4.12}$$

While, for the first term of (3.4.12), using the Lemma 2.1 in [**Po1**], we have

$$\int_0^T \int_\Omega (v^\epsilon)_t w^\epsilon \ dxdt \ge \omega(\epsilon, \mu, h).$$

Hence (3.4.12) becomes

$$\int\int_{\{|v^\epsilon| \le k\}} a(t, x, \nabla u^\epsilon) \cdot \nabla (v^\epsilon - (T_k(v))_\mu) \ dxdt \le \omega(\epsilon, \mu, h). \tag{3.4.13}$$

While, since $\nabla T_k(v)_\mu \to \nabla T_k(v)$ strongly in $(L^{p(\cdot)}(Q))^N$ as $\mu \to +\infty$ and $g_2^\epsilon \to g_2$ strongly in $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, thanks to (3.4.13), we easily obtain

$$\int_0^T \int_\Omega a(t, x, \nabla (g_2^\epsilon + T_k(v^\epsilon)) \chi_{\{|v^\epsilon| \le k\}}) \cdot \nabla (u^\epsilon - T_k(v))) dxdt.$$

Moreover, again thanks to the fact that $\nabla T_k(v)_\mu \to \nabla T_k(v)$ strongly in $(L^{p(\cdot)}(Q))^N$ as $\mu \to +\infty$, and from (3.4.13),

$$\int_0^T \int_\Omega a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le k\}}) \cdot \nabla (T_k(v^\epsilon) - T_k(v)) dxdt \le \omega(\epsilon, \mu, h).$$

Therefore, passing to the limit in (3.4.11) as ϵ tends to zero, μ and h tends to infinity respectively, we deduce that

$$\limsup_{\epsilon \to 0} \int_0^T \int_\Omega a(t, x, \nabla u^\epsilon \chi_{\{|v^\epsilon| \le k\}}) \cdot \nabla (T_k(v^\epsilon) - T_k(v)) \le 0.$$

Now, let k be such that $\chi_{\{|v^\epsilon| \le k\}} \to \chi_{\{|v| \le k\}}$ a.e. and $g_2^n \to g_2$ strongly in $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, then using (3.2.2) and Lemma 3.2 in [**B**], we get

$$a(t, x, \nabla (g_2^n + T_k(v) \chi_{\{|v^\epsilon| \le k\}})) \to a(t, x, \nabla (g_2 + T_k(v) \chi_{\{|v| \le k\}})) \text{ in } (L^{p(\cdot)}(Q))^N \tag{3.4.14}$$

and from (3.4.14) we obtain

$$\begin{aligned}&\int_0^T\int_\Omega(a(t,x,\nabla(g_2^n+T_k(v^\epsilon)))-a(t,x,\nabla(g_2+T_k(v))))\cdot\nabla(T_k(v^\epsilon)-T_k(v))dxdt\\&\leq-\int_0^T\int_\Omega a(t,x,\nabla(g_2+T_k(v)))\cdot\nabla(T_k(v^\epsilon)-T_k(v))dxdt+\omega(\epsilon,\mu,h).\end{aligned}\tag{3.4.15}$$

When we use the weak convergence of $\nabla T_k(v^\epsilon)$ to $\nabla T_k(v)$ in $(L^{p(\cdot)}(Q))^N$, we can conclude that

$$\limsup_{\epsilon\to0}\int_0^T\int_\Omega a(t,x,\nabla(g_2^\epsilon+T_k(v))\chi_{\{|v^\epsilon|\leq k\}})\cdot\nabla(T_k(v^\epsilon)-T_k(v))dxdt=0.$$

In the same time, we can pass to the limit in (3.4.15) as ϵ tends to zero, μ and h tends to infinity respectively, to deduce that

$$\limsup_{\epsilon\to0}\int_0^T\int_\Omega[a(t,x,\nabla u^\epsilon\chi_{\{|v^\epsilon|\leq k\}})-a(t,x,\nabla(g_2^\epsilon+T_k(v))\chi_{\{|v^\epsilon|\leq k\}})]\cdot(\nabla u^\epsilon-\nabla(g_2^\epsilon+T_k(v)))dxdt=0.$$

Using that $\chi_{\{|v^\epsilon|\leq k\}}$ converges a.e. to $\chi_{\{|v^\epsilon|\leq k\}}$ and that g_2^ϵ strongly converges to g_2 in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, then thanks to the standard monotonicity argument which relies on (3.2.3) (see Lemma 5 in **[BMP]**) we readily have from (3.4.16),

$$\nabla u^\epsilon\chi_{\{|v^\epsilon|\leq k\}}\to\nabla(g_2+T_k(v))\chi_{\{|v^\epsilon|\leq k\}}=\nabla u\chi_{\{|v^\epsilon|\leq k\}}\text{ a.e. in }Q,$$

which means that

$$a(t,x,\nabla u^\epsilon\chi_{\{|v^\epsilon|\leq k\}})\cdot\nabla u^\epsilon\to a(t,x,\nabla u\chi_{\{|v^\epsilon|\leq k\}})\cdot\nabla u\text{ strongly in }L^1(Q)\text{ and a.e. in }Q.$$

Finally, collecting together all these facts with (3.2.1), we obtain the equi-integrability of the sequence $|\nabla u^\epsilon|^{p(x)}\chi_{\{|v^\epsilon|\leq k\}}$ in Q, we can write as consequences of Vitali's theorem and since g_2^ϵ strongly converges in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ yields

$$T_k(u^\epsilon-g_2^\epsilon)\to T_k(u-g_2)\text{ strongly in }L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)).$$

Now, we have to check that

$$\nabla T_k(u^\epsilon-g_2^\epsilon)\to\nabla T_k(u^\epsilon-g_2^\epsilon)\text{ in }(L^{p(\cdot)}(Q))^N.$$

We need the following lemmas.

LEMMA 3.28. *Let $v,v_n\in L^{p(\cdot)}(Q)$, $n=1,2,\cdots$. Then the following statements are equivalent*

(1) $\lim_{n\to\infty}|v_n-v|_{\rho(\cdot)}=0$,

(2) $\lim_{n\to\infty}(v_n-v)=0$,

(3) v_n converges to v in Q in measure and $\lim_{n\to\infty}\rho_{p(\cdot)}(v_n)=\rho_{p(\cdot)}(v)$.

PROOF. See **[FZ1]**, Theorem 1.4. □

LEMMA 3.29. *(Lebesgue Generalized Convergence Theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and f a measurable function such that $f_n\to f$ a.e. in Q. let $(g_n)_{n\in\mathbb{N}}\subset L^1(Q)$ such that for all $n\in\mathbb{N}$, $|f_n|\leq g_n$ a.e. in Q and $g_n\to g$ in $L^1(Q)$. Then*

$$\int_Q f_ndxdt\to\int_Q fdxdt.$$

Now, set $f^\epsilon=|\nabla T_k(u^\epsilon)|^{p(x)}$, $f=|\nabla T_k(u)|^{p(x)}$, $g^\epsilon=a(t,x,\nabla u^\epsilon\chi_{\{|v^\epsilon|\leq k\}})\cdot\nabla u^\epsilon$ and $g=a(t,x,\nabla u\chi_{\{|v^\epsilon|\leq k\}})\cdot\nabla u$, f^ϵ is a sequence of measurable functions, f is a measurable function and according to the almost convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ in Ω,

$$f^\epsilon\to f\text{ a.e. in Q.}$$

Using $a(x,\nabla T_k(u^\epsilon))\cdot\nabla T_k(u^\epsilon)\to a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ strongly in $L^1(\Omega)$ and a.e. in Ω, we have $(g^\epsilon)_{\epsilon>0}\subset L^1(Q)$, $g^\epsilon\to g$ a.e. in Q, $g^\epsilon\to g$ in $L^1(Q)$, and $|f^\epsilon|\leq Cg^\epsilon$. Then, by Lemma 3.29 we have

$$\int\int_Q f^\epsilon dxdt\to\int\int_Q fdxdt,$$

which is equivalent to

$$\int\int_Q |\nabla T_k(u^\epsilon)|^{p(x)} dxdt \to \int\int_Q |\nabla T_k(u)|^{p(x)} dxdt.$$

We deduce from (2) that the sequence $(\nabla T_k(u^\epsilon))_{\epsilon>0}$ converges to $\nabla T_k(u)$ in Q in measure. Then, by Lemma 3.28, we deduce that

$$\lim_{\epsilon\to 0}\int\int_Q |\nabla T_k(u^\epsilon) - \nabla T_k(u)|^{p(x)} dxdt = 0,$$

which is equivalent to saying that

$$\nabla T_k(u^\epsilon) \to \nabla T_k(u) \text{ in } (L^{p(\cdot)}(Q))^N.$$

Finally, we are able to prove that problem (3.2.4) has a renormalized solution. Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support, and let $\varphi \in C_c^\infty(Q)$; then the approximating solutions u^ϵ and $u^\epsilon - g_2^\epsilon$ satisfy

$$\begin{aligned}
&-\int_\Omega S(u_0^\epsilon)\varphi(0)dx - \int_0^T \langle \varphi_t, S(u^\epsilon - g_2^\epsilon)\rangle + \int_Q S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla\varphi\, dxdt \\
&+\int_Q S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon)\varphi\, dxdt \\
&= \int_Q S'(u^\epsilon - g_2^\epsilon)f^\epsilon\varphi\, dxdt + \int_Q S'(u^\epsilon - g_2^\epsilon)F^\epsilon\cdot\nabla\varphi\, dxdt + \int_Q S''(u^\epsilon - g_2^\epsilon)F^\epsilon\cdot\nabla(u^\epsilon - g_2^\epsilon)\varphi\, dxdt \\
&+\int_Q S'(u^\epsilon - g_2^\epsilon)G_1^\epsilon\cdot\nabla\varphi\, dxdt + \int_Q S''(u^\epsilon - g_2^\epsilon)G_1^\epsilon\cdot\nabla(u^\epsilon - g_2^\epsilon)\varphi\, dxdt.
\end{aligned}\tag{3.4.16}$$

We consider the first term in the left-hand side of (3.4.16). Since S is continuous, Proposition 3.27 implies that $S(u^\epsilon - g_2^\epsilon)$ converges to $S(u - g_2)$ a.e. in Q and weakly–* in $L^\infty(Q)$. Then $(S(u^\epsilon - g_2^\epsilon))_t$ converges to $(S(u - g_2))_t$ in $D'(Q)$ as $\epsilon \to 0$, that is

$$\int_0^T\int_\Omega (S(u^\epsilon - g_2^\epsilon))_t\varphi\, dxdt \to \int_0^T\int_\Omega (S(u - g_2))_t\varphi\, dxdt.$$

As supp $S' \subset [-M, M]$ for some $M > 0$, we have

$$S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon) = S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon))$$

and

$$S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon) = S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon)\cdot\nabla T_M(u^\epsilon - g_2^\epsilon).$$

Using Proposition 3.27, the strong convergence of g_2^ϵ to g_2 in $L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ and assumption (3.2.2), we have

$$S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon) \to S'(u - g_2)a(t,x,\nabla T_M(u - g_2) + \nabla g_2) \text{ in } (L^{p'(\cdot)}(Q))^N$$

and

$$S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon)\cdot\nabla T_M(u^\epsilon - g_2^\epsilon) \to S''(u - g_2)a(t,x,\nabla T_M(u - g_2) + \nabla g_2)\cdot\nabla T_M(u - g_2)$$

in $L^1(Q)$. The pointwise convergence of $S'(u^\epsilon - g_2^\epsilon)$ to $S'(u - g_2)$ and the strong convergence of f^ϵ to f in $L^1(Q)$ yield

$$f^\epsilon S'(u^\epsilon - g_2^\epsilon) \to f S'(u - g_2) \text{ strongly in } L^1(Q) \text{ as } \epsilon \to 0.$$

Finally, we recall that $\nabla S'(u^\epsilon - g_2^\epsilon) \to \nabla S'(u - g_2)$ weakly in $(L^{p(\cdot)}(Q))^N$. Then the term $S''(u^\epsilon - g_2^\epsilon)\, F\cdot\nabla(u^\epsilon - g_2^\epsilon)$ which is equal to $F\cdot\nabla S'(u^\epsilon - g_2^\epsilon)$ verifies the following convergence result.

$$S''(u^\epsilon - g_2^\epsilon)F\cdot\nabla(u^\epsilon - g_2^\epsilon) \rightharpoonup F\cdot\nabla S'(u - g_2) \text{ in } L^1(Q) \text{ as } \epsilon \to 0.$$

We can identifies the term $F\cdot\nabla S'(u - g_2)$ with $S''(u - g_2)\, F\cdot\nabla(u - g_2)$. As a consequence of the last convergence results, we are in position to pass to the limit as $\epsilon \to 0$ in (3.4.16), and to conclude that u satisfies Definition 3.21. It remains to show that $S(u - g_2)$ satisfies the initial condition (3.3.5). To this end, we take in mind the last convergence results of the terms of equation (3.4.16), which imply that

$$(S(u^\epsilon - g_2^\epsilon))_t \text{ is bounded in } X^* + L^1(Q).$$

While $S(u^\epsilon - g_2^\epsilon)$ strongly converges in X, we deduce, see **[Po1]**, Theorem 1.1, that $S(u^\epsilon - g_2^\epsilon)$ being bounded in $L^\infty(Q)$ and

$$S(u^\epsilon - g_2^\epsilon) \to S(u - g_2) \text{ strongly in } C([0,T]; L^1(Q)).$$

It follows that

$$S(u^\epsilon - g_2^\epsilon)(0) \to S(u_0) \text{ strongly in } L^1(Q).$$

Hence (3.3.5) fulfilled. Thus, the proof of existence of renormalized solution u of problem (3.2.4) is complete.

Now, we try to stress the fact that the notion of renormalized solution should be the right one to get uniqueness by choosing an appropriate test function motivated by **[BW]**. Let S_n be defined as in Definition 3.21. We take $T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))$ as a test function in both the equation solved by u_1 and u_2 and subtract them to obtain that

$$\mathcal{J}_0 + \mathcal{J}_1 = \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6 + \mathcal{J}_7,$$

where

$$\mathcal{J}_0 = \int_0^T \int_\Omega (S_n(u_1 - g_2) - S_n(u_2 - g_2))_t T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt,$$

$$\mathcal{J}_1 = \int_0^T \int_\Omega [S_n'(u_1 - g_2)a(t,x,\nabla u_1) - S_n'(u_2 - g_2)a(t,x,\nabla u_2)] \cdot \nabla T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt,$$

$$\mathcal{J}_2 = -\int_Q [S_n''(u_1 - g_2)a(t,x,\nabla u_1) \cdot \nabla(u_1 - g_2) - S_n''(u_2 - g_2)a(t,x,\nabla u_2) \cdot \nabla(u_2 - g_2)] \\ \cdot [T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt],$$

$$\mathcal{J}_3 = \int_Q f(S_n'(u_1 - g_2) - S_n'(u - g_2))T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt,$$

$$\mathcal{J}_4 = \int_Q F(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \cdot \nabla T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt,$$

$$\mathcal{J}_5 = \int_Q [S_n''(u_1 - g_2)F \cdot \nabla(u_1 - g_2) - S_n''(u_2 - g_2)F \cdot \nabla(u_2 - g_2)]T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt,$$

$$\mathcal{J}_6 = \int_Q [G_1(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \cdot \nabla T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt],$$

$$\mathcal{J}_7 = \int_Q [S_n''(u_1 - g_2)G_1 \cdot \nabla(u_1 - g_2) - S_n''(u_2 - g_2)G_1 \cdot \nabla(u_2 - g_2)]T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt.$$

We estimate $\mathcal{J}_i$, $i = 1, \cdots, 7$ one by one. Recalling the definition of $\Theta_k(r)$, J_0 can be written as

$$\mathcal{J}_0 = \int_\Omega \Theta_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))(T)dx - \int_\Omega \Theta_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))(T)dx.$$

Due to the same initial condition for $u_1 - g_2$ and $u_2 - g_2$, and the properties of Θ_k, we get

$$\mathcal{J}_0 = \int_\Omega \Theta_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))(0)dx \geq 0.$$

We deal with $\mathcal{J}_1$ splitting it as bellow

$$\begin{aligned}
\mathcal{J}_1 = &\int\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\leq k\}\cap\{|u_1-g_1|\leq n, |u_2-g_2|\leq n\}} [a(t,x,\nabla u_1) - a(t,x,\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2)dxdt \\
&+ \int\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\leq k\}\cap\{|u_1-g_2|\leq n, |u_2-g_2|>n\}} [S_n'(u_1 - g_2)a(t,x,\nabla u_1) - S_n'(u_2 - g_2)a(t,x,\nabla u_2)] \\
&\qquad \cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))]dxdt \\
&+ \int\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\leq k\}\cap\{|u_2-g_2|>n\}} [S_n'(u_1 - g_2)a(t,x,\nabla u_1) - S_n'(u_2 - g_2)a(t,x,\nabla u_2)] \\
&\qquad \cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))]dxdt \\
:= &\ \mathcal{J}_1^1 + \mathcal{J}_1^2 + \mathcal{J}_1^3.
\end{aligned}$$

Next, as $\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\leq k,\ |u_1-g_2|>n\}\subset\{|u_1-g_2|>n,\ |u_2-g_2|>n-k\}$ and using the fact that $S_n'(t)=0$ if $|t|>n+1$ and $|S_n'(t)|\leq 1$, we have

$$
\begin{aligned}
|\mathcal{J}_1^3| \leq & \int\int_{\{n\leq|u_2-g_2|\leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt \\
& + \int\int_{\{n\leq|u_1-g_2|\leq n+1\}\cap\{n-k\leq|u_2-g_2|\leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_2-g_2)|dxdt \\
& + \int\int_{\{n\leq|u_1-g_2|\leq n+1\}\cap\{n-k\leq|u_2-g_2|\leq n+1\}} |a(t,x,\nabla u_2)||\nabla(u_1-g_2)|dxdt \\
& + \int\int_{\{n-k\leq|u_2-g_2|\leq n+1\}} |a(t,x,\nabla u_2)||\nabla(u_2-g_2)|dxdt.
\end{aligned}
\tag{3.4.17}
$$

We deduce from the first integral in the right- hand side of (3.4.17),

$$
\begin{aligned}
& \int\int_{\{n\leq|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt \\
& \leq \int\int_{\{n\leq|u_1-g_2|\leq n+1\}} \beta(b(t,x)+|\nabla u_1|^{p(x)-1})|\nabla(u_1-g_2)|dxdt \\
& \leq \int\int_{\{n\leq|u_1-g_2|\leq n+1\}} \beta b(t,x)|\nabla(u_1-g_2)|dxdt + \int\int_{\{n\leq|u_1-g_2|\leq n+1\}} \beta|\nabla u_1|^{p(x)-1}|\nabla(u_1-g_2)|dxdt \\
& \leq \int\int_{\{|u_1-g_2|\leq n+1\}} \frac{C}{p'_-}|b(t,x)|^{p'(x)}dxdt + \int\int_{\{|u_1-g_2|\leq n+1\}} \frac{C}{p_-}|\nabla(u_1-g_2)|^{p(x)}dxdt \\
& + \int\int_{\{|u_1-g_2|\leq n+1\}} \frac{C}{p'_-}|\nabla u_1|^{p(x)}dxdt + \int\int_{\{|u_1-g_2|\leq n+1\}} \frac{C}{p_-}|\nabla(u_1-g_2)|^{p(x)}dxdt.
\end{aligned}
$$

Since $b(t,x)$ is bounded in $L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $\text{meas}\{n\leq|u_1-g_2|\leq n+1\}$ converges uniformly to zero as n tends to infinity, we deduce from the conditions (3.3.3) and (3.3.7) that

$$
\lim_{n\to+\infty}\int\int_{\{|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt = 0.
$$

Similarly, we prove that all the other integrals in the right-hand side of (3.4.17) converge to zero as $n\to+\infty$. Thus $\mathcal{J}_1^3$ converges to zero. Changing the roles of u_1-g_2 and u_2-g_2, we may get the similar arguments for $\mathcal{J}_1^2$. Furthermore, $\mathcal{J}_1^2$ converges to 0. An application of Fatou's Lemma gives

$$
\liminf_{n\to+\infty}\mathcal{J}_1 \geq \int\int_{\{|u_1-u_2|\leq k\}} [a(t,x,\nabla u_1)-a(t,x,\nabla u_2)]\cdot(\nabla u_1-\nabla u_2)dxdt.
$$

Now, we can pass to the study of the limit of $\mathcal{J}_2$. We have

$$
\begin{aligned}
\mathcal{J}_2 = & \int_0^T\int_\Omega [S_n''(u_1-g_2)a(t,x,\nabla u_1)\cdot\nabla(u_1-g_2)]T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt \\
& + \int_0^T\int_\Omega [S_n''(u_2-g_2)a(t,x,\nabla u_2)\cdot\nabla(u_2-g_2)]T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt \\
= & \mathcal{J}_2^1+\mathcal{J}_2^2.
\end{aligned}
$$

By symmetry between $\mathcal{J}_2^1$ and $\mathcal{J}_2^2$, it is enough to prove that J_2^1 tends to 0.
Since $|S_n''(s)|\leq 1$ and $S_n''(s)\neq 0$ only if $|s|\in[n,n+1]$, using (3.2.2) we can write

$$
\begin{aligned}
|\mathcal{J}_2^1| & \leq k\int\int_{\{n\leq|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_1-g_2)| \\
& \leq k\int_{\{n\leq|u_1-g_2|\leq n+1\}} \beta(b(t,x)+|\nabla u_1|^{p(x)-1})||\nabla(u_1-g_2)|dxdt \\
& \leq k\int_\Omega \beta(b(t,x)+|\nabla u_1|^{p(x)-1})|\nabla(u_1-g_2)|\chi_{\{n\leq|u_1-g_2|\leq n+1\}}dxdt \\
& \to 0 \text{ as } n\to+\infty.
\end{aligned}
$$

We conclude that

$$\lim_{n\to+\infty} \mathcal{J}_2 = 0.$$

Let us recall that by definition of S_n, we have that S_n' converge to 1 for every s in $\mathbb{R}$. Then

$$f(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \to 0 \text{ strongly in } L^1(Q) \text{ as } n \to +\infty.$$

Using the dominated convergence Theorem, we deduce that

$$\lim_{n\to+\infty} \mathcal{J}_3 = 0.$$

Let us study the limit of $\mathcal{J}_6$, we have $S_n'(u_1 - g_2) - S_n'(u_2 - g_2) = 0$ in $\{|u_1 - g_2| \le n, |u_2 - g_2| \le n\} \cup \{|u_1| > n+1, |u_2| > n+1\}$, then

$$\mathcal{J}_6 = \mathcal{J}_6^1 + \mathcal{J}_6^2 + \mathcal{J}_6^3,$$

where

$$\begin{aligned}\mathcal{J}_6^1 = \int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\le k\}\cap\{|u_1-g_2|\le n, |u_2-g_2|>n\}} & [G_1(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \\ & \cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))]\end{aligned}$$

Recalling that $S_n(t) = t$ if $|t| \le n$, S_n is nondecreasing and Supp $S_n' \subset [-n-1, n+1]$, we have

$$|\mathcal{J}_6^1| \le \int_{\{n-k\le|u_1-g_2|\le n\}} |G_1||\nabla(u_1 - g_2)|dxdt + \int_{\{n\le|u_2-g_2|\le n+1\}} |G_1||\nabla(u_2 - g_2)|dxdt.$$

So that, using Hölder's inequality, we get

$$\begin{aligned}|\mathcal{J}_6^1| \le & C\|G_1\|_{p'(x)} (\max(\int_{\{n-k\le|u_1-g_2\le n|\}} |\nabla u_1 - \nabla g_2|^{p(x)})^{\frac{1}{p_-}}, (\int_{\{n-k\le|u_1-g_2|\le n\}} |\nabla u_1 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p_+}}) \\ & + \max(\int_{\{n\le|u_2-g_2|\le n+1\}} |\nabla u_2 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p_-}}, (\int_{\{n\le|u_2-g_2|\le n+1\}} |\nabla u_2 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p_+}}))\end{aligned}$$

Thus by (3.3.3) we get that $(\mathcal{J}_6^1)$ converges to zero as n tends to infinity.
The same is true for $(\mathcal{J}_6^2)$.

$$\begin{aligned}\mathcal{J}_6^2 = \int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\le k\}\cap\{n\le|u_1-g_2|\le n+1\}} & [G_1(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \\ & \cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt].\end{aligned}$$

Since $|S_n(t)| > n - k$ implies $|t| > n - k$, we have

$$|\mathcal{J}_6^2| \le \int_{\{n\le|u_1-g_2|\le n+1\}} |G_1||\nabla(u_1 - g_2)|dxdt + \int_{\{n-k\le|u_2-g_2|\le n+1\}} |G_1||\nabla(u_2 - g_2)|dxdt.$$

So that using Hölder's inequality and (3.3.3), we get that $(\mathcal{J}_6^2)$ converges to zero as n tends to infinity.
The term $(\mathcal{J}_6^3)$ can be dealt with the same way using that $S_n'(t) = 0$ if $|t| > n + 1$. Hence we deduce

$$\lim_{n\to+\infty} \mathcal{J}_6 = 0.$$

As regards $(\mathcal{J}_7)$, note that using the properties of S_n'' and (3.2.2), we can split the integral as follows

$$\begin{aligned}|\mathcal{J}_7| = & \int_Q S_n''(u_1 - g_2)G_1 \cdot \nabla(u_1 - g_2)T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt \\ & - \int_Q S_n''(u_2 - g_2)G_1 \cdot \nabla(u_2 - g_2)T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt.\end{aligned} \tag{3.4.18}$$

We denote $(\mathcal{J}_7^1, \mathcal{J}_7^2)$ the two integrals of (3.4.18). Using the properties of S_n and S_n'' (recall that $S_n''(s) = -\mathrm{sgn}(s)\chi_{\{n\le|s|\le n+1\}}$) we have

$$\begin{aligned}
|\mathcal{J}_7^1| &\le k\int_{\{n\le|u_1-g_2|\le n+1\}} |G_1||\nabla(u_1-g_2)|dxdt\\
&\le Ck\|G_1\|_{L^{p'(x)}(Q)}\\
&\quad\times \max\Big(\int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla u_1-\nabla g_2|^{p(x)}dxdt)^{\frac{1}{p_-}}, (\int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla u_1-\nabla g_2|^{p(x)}dxdt)^{\frac{1}{p_+}}\Big).
\end{aligned}$$

Applying Hölder inequality and using property (3.3.7), we easily get that $(\mathcal{J}_7^1)$ converges to zero as n tends to infinity. Similarly, we have

$$\begin{aligned}
|\mathcal{J}_7^2| &\le Ck\|G_1\|_{L^{p'(x)}(Q)}\\
&\quad\times \max\Big(\int_{\{n\le|u_2-g_2|\le n+1\}} |\nabla u_2-\nabla g_2|^{p(x)}dxdt)^{\frac{1}{p_-}}, (\int_{\{n\le|u_2-g_2|\le n+1\}} |\nabla u_2-\nabla g_2|^{p(x)}dxdt)^{\frac{1}{p_+}}\Big).
\end{aligned}$$

Again Hölder inequality together with (3.3.3) allow to deduce that $(\mathcal{J}_7^2)$ converges to zero as well. So that we finally get that

$$\lim_{n\to+\infty} \mathcal{J}_7 = 0.$$

Similarly we have

$$\lim_{n\to+\infty} \mathcal{J}_4 = 0 \text{ and } \lim_{n\to+\infty} \mathcal{J}_5 = 0.$$

Putting together $(\mathcal{J}_1 - \mathcal{J}_6)$ and $(\mathcal{J}_7)$, we obtain $\lim\limits_{n\to\infty} \sum_{i=0}^{1} \mathcal{J}_i = \lim\limits_{n\to\infty} \sum_{i=2}^{7} \mathcal{J}_i$, as n tends to infinity. Then

$$\int_{\{|u_1-u_2|\le k\}} [a(t,x,\nabla u_1) - a(t,x,\nabla u_2)]\cdot(\nabla u_1 - \nabla u_2)dxdt \le 0,$$

and letting k tends to infinity (recall that u_1 and u_2 are finite a.e. in Q), we deduce that

$$\int_Q [a(t,x,\nabla u_1) - a(t,x,\nabla u_2)]\cdot(\nabla u_1 - \nabla u_2)dxdt \le 0.$$

The strict monotonicity assumption (3.2.3) implies that $\nabla u_1 = \nabla u_2$ a.e. in Q.
Then, let $\zeta_n = T_1(T_{n+1}(u_1-g_2) - T_{n+1}(u_2-g_2))$. We have $\zeta_n \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and since $\nabla(u_1-g_2) = \nabla(u_2-g_2)$ a.e. in Q,

$$\nabla\zeta_n = \begin{cases} 0 \quad \text{on } \{|u_1-g_2|\le n+1, |u_2 -_2| \le n+1\} \cup \{|u_1-g_2| > n+1, |u_2-g_2| > n+1\},\\ \chi_{\{u_1-g_2-T_{n+1}(u_2-g_2)|\le 1\}}\nabla(u_1-g_2) \quad \text{on } \{|u_1-g_2|\le n+1, |u_2-g_2|>n+1\},\\ -\chi_{\{u_2-g_2-T_{n+1}(u_1-g_2)|\le 1\}}\nabla(u_2-g_2) \quad \text{on } \{|u_1-g_2|>n+1, |u_2-g_2|\le n+1\}.\end{cases}$$

But, if $|s| > n+1, |t| \le n+1$ and $|t - T_{n+1}(s)| \le 1$, then $n \le |t| \le n+1$, which implies that

$$\begin{aligned}
\int_Q |\nabla\zeta_n|^{p(x)}dxdt &\le \int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla(u_1-g_2)|^{p(x)}dxdt + \int_{\{n\le|u_2-g_2|\le n+1\}} |\nabla(u_2-g_2)|^{p(x)}dxdt.\\
&\to 0 \text{ as } n\to+\infty
\end{aligned}$$

Then, $\zeta_n \to 0$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, and thus in $\mathcal{D}'(Q)$ as $n\to+\infty$. Since $\zeta_n \to T_1(u_1-g_1) - (u_2-g_2))$ a.e. in S as $n\to+\infty$ and remains bounded by 1, we also have $\zeta_n \to T_1((u_1-g_2)-(u_2-g_2))$ in $\mathcal{D}'(Q)$. Hence, $T_1((u_1-g_2)-(u_2-g_2)) = 0$ i.e., $u_1 - g_2 = u_2 - g_2$ on Q. Therefore $u_1 = u_2$. Thus, we obtain the uniqueness of the renormalized solution to (3.2.4). □

CHAPTER 4

Nonlinear parabolic problems with Leray–Lions operators and general measure data

In this Chapter, the starting point will be the end of the first point of the proof in [**Pe1**] (the a priori estimates), and the goal will be to pass to the limit in ϵ using the equation solved by u_ϵ (see (4.0.1)). The major advantage of our approach is that we can perform the passage to the limit using the almost everywhere convergence of the gradients in Q. In the proof of Theorem 2 in [**Pe1**], the author used the fact that the approximating sequences μ_ϵ having a splitting converging to μ, the estimate concerning u_ϵ and $u_\epsilon - g_\epsilon^t$, next he prove the strong convergence of $T_k(u_\epsilon - g_\epsilon)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. To obtain this result, he use the same technique as in [**DMOP**] adapted to the parabolic case. In the present Chapter we generalize this existence result to renormalized solutions of problems depending on u and ∇u using a new proof of the almost everywhere convergence of gradients

$$\begin{cases} (u_\epsilon)_t - \operatorname{div}(a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)) = \mu_\epsilon & \text{in } Q := (0,T)\times\Omega, \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_\epsilon(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.0.1}$$

where (μ_ϵ) is a sequences of measures with splitting converging to μ, and

$$\lim_{\epsilon\to 0} a_\epsilon(t,x,s_\epsilon,\zeta_\epsilon) = a_0(t,x,s,\zeta),$$

for every sequence $(s_\epsilon,\zeta_\epsilon) \in \mathbb{R}\times\mathbb{R}^N$ converging to (s,ζ) and for a.e. $(t,x)\in Q$. The proof in this chapter is rather technical, and it can be split into two parts. As a first step, the equation solved by u_ϵ is used in order to obtain some a priori estimates, and hence a weak limit u of (u_ϵ), which is the candidate to be the solution to (4.1.1). In particular it is easily proved that, up to a subsequence, every truncation $T_k(u_\epsilon)$ converges to the corresponding truncation $T_k(u)$ in the weak topology of $L^p(0,T;W_0^{1,p}(\Omega))$. The second part, which is the hardest one, is devoted to showing that the sequence of truncations converges, in fact, in the strong topology of $L^p(0,T;W_0^{1,p}(\Omega))$. The main point which allows to go further the previous works, is the proof of the almost everywhere convergence of gradients in Proposition 4.16 using the technique developed in [**Pr1, Po1**]. In order to underline the importance of this tool, we have chosen to plan this Chapter in the following way. In Section 4.1, we recall some basic assumptions and notations and we introduce the definition of renormalized solutions. In Section 4.2, we investigate the link between measures in Q and the notion of parabolic capacity, this notion can be obtained from the result of the "elliptic capacity" contained in [**D**], which can be slightly adapted to this context of parabolic spaces, we show the decomposition method for more general measures with bounded total variation in order to find a sense of solution to Cauchy-Dirichlet problems, and we introduce and study a special type of approximating sequences of measures obtained via convolution arguments. In Section 4.3, we establish the fundamental a priori estimates and we prove convergence results to limit functions. Finally, in Section 4.4 we show the interest of cut-off functions and intermediary lemmas to prove the strong convergence of truncates and to obtain the main result.

4.1. Assumptions on the operator and renormalized formulation

Throughout this Chapter Ω will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, p and p' will be real numbers, with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, $|\zeta|$ and $\zeta \cdot \zeta'$ will denote respectively the Euclidean norm of a vector $\zeta \in \mathbb{R}^N$ and the scalar product between ζ and $\zeta' \in \mathbb{R}^N$. Consider the parabolic problem

$$\begin{cases} u_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu & \text{in } Q := (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.1.1}$$

where $T > 0$, Q is the cylinder $(0,T)\times\Omega$, $(0,T)\times\partial\Omega$ being its lateral surface, the operator of Leray-Lions $u \mapsto -\operatorname{div}(a(t,x,u,\nabla u))$ is pseudo-monotone defined on the space $L^p(0,T;W_0^{1,p}(\Omega))$ with values in its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We assume that $u_0 \in L^2(\Omega)$ and the data μ is a Radon measure with bounded variation on Q. Fixed three positive constants c_0, c_1, c_2, and a non-negative function $b_0 = b(t,x) \in L^{p'}(Q)$, we say that a function $a : (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions $H(c_0,c_1,c_2,b_0)$ if a is a Carathéodory function (that is, $a(\cdot,\cdot,s,\zeta)$ is measurable on Q for every (s,ζ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every (t,x) in Q) such that, for every $s \in \mathbb{R}$, $\zeta,\zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$, satisfying the following properties.

$$a(t,x,s,\zeta)\cdot\zeta \geq c_0|\zeta|^p, \tag{4.1.2}$$

$$|a(t,x,s,\zeta)| \leq b_0(t,x) + c_1|s|^{p-1} + c_2|\zeta|^{p-1}, \tag{4.1.3}$$

$$(a(t,s,s,\zeta) - a(t,x,s,\zeta'))\cdot(\zeta-\zeta') > 0. \tag{4.1.4}$$

Notice that, as a consequence of (4.1.2) and of the continuity of a with respect to ζ, we have that $a(t,x,s,0) = 0$ for a.e. (t,x) in Q and for every $s \in \mathbb{R}$. Thanks to assumptions $H(c_0,c_1,c_2,b_0)$, the map $u \mapsto -\operatorname{div}(a(t,x,u,\nabla u))$ is a coercive, continuous, bounded and monotone operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ with values into its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$; hence by the standard theory of monotone operators (see [**L**]), for every F in $L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$ there exists a variational solution u of the problem

$$\begin{cases} u_t - \operatorname{div}(a(t,x,v,\nabla v)) = F & \text{in } Q := (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ v(0) = u_0 & \text{in } \Omega, \end{cases}$$

in the sense that v belongs to $W \cap C(0,T;L^2(\Omega))$ (where $W = \{u \in L^p(0,T;V),\ u_t \in L^{p'}(0,T;V')\}$ with $V = W_0^{1,p}(\Omega)\cap L^2(\Omega)$), and

$$-\int_\Omega u_0\varphi(0)\,dx - \int_0^T \langle\varphi_t, v\rangle\,dt + \int_Q a(t,x,v,\nabla v)\cdot\nabla\varphi\,dx\,dt = \int_0^T \langle F,\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}dt, \tag{4.1.5}$$

for all $\varphi \in W$ such that $\varphi(T) = 0$. (Here and in the following $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$). For any $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ (see Figure 1) by

$$T_k(t) = \max(-k, \min(k,t)), \quad t \in \mathbb{R}$$

Let us consider the space of all measurable functions, finite a.e. in Q such that $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$.

We can see that every function u in this space has a cap_p quasi-continuous representative, that will always be identified with u. Moreover, there exists a measurable function $v : Q \to \mathbb{R}^N$, which is unique up to almost everywhere equivalence, such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ a.e. in Q, for every $k > 0$, (see [**B6**, Lemma 2.1]). Hence it is possible to define a generalized gradient ∇u of u, setting $\nabla u = v$. If $u \in L^1(0,T;W_0^{1,1}(\Omega))$, this gradient coincide with the usual gradient in distributional sense. In the sequel we suppose that p satisfies $p > 2 - \frac{1}{N+1}$. Then the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is valid, i.e.,

$$X = L^p((0,T);W_0^{1,p}(\Omega)), \quad X' = L^{p'}((0,T);W^{-1,p'}(\Omega)).$$

Let $T_k(t)$ be the Lipschitz continuous function $T_k : \mathbb{R} \to \mathbb{R}$, so that we recall the auxiliary functions

$$\Theta_n(s) = T_1(s - T_n(s)),\ h_n(s) = 1 - (\Theta_n(s)),\ S_n(s) = \int_0^s h_n(r)dr,\ \forall s \in \mathbb{R},$$

defined in Figures 6–8. We are now in a position to introduce (following [**Pe1**]) the notion of renormalized solution. To simplify the notation, let us define $v = u - g$, where u is the solution and g is the time-derivative part of μ_0, and $\hat{\mu}_0 = \mu - g_t - \mu_s = f - \operatorname{div}(G)$.

DEFINITION 4.1. Let $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_b(Q)$. A measurable function u is a renormalized solution of problem (4.1.1) if there exists a decomposition (f, G, g) of μ_0 such that

$$v = u - g \in L^q(0,T; W_0^{1,q}(\Omega)) \cap L^\infty(0,T; L^1(\Omega)) \quad \forall q < p - \frac{N}{N+1}, \qquad T_k(v) \in X \quad \forall k > 0, \tag{4.1.6}$$

and, for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} &-\int_\Omega S(u_0)\varphi(0)\,dx - \int_0^T \langle \varphi_t, S(v)\rangle\,dt + \int_Q S'(v) a(t,x,u,\nabla u)\cdot\nabla\varphi\,dx\,dt \\ &+ \int_Q S''(v) a(t,x,u,\nabla u)\cdot\nabla v\varphi\,dx\,dt = \int_Q S'(v)\varphi\,d\tilde{\mu}_0, \end{aligned} \tag{4.1.7}$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(\cdot, T) = 0$; for any $\psi \in C(\overline{Q})$

$$\begin{aligned} &\lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le v<2n\}} a(t,x,u,\nabla u)\cdot\nabla v\psi\,dx\,dt = \int_Q \psi d\mu_s^+, \\ &\lim_{n\to+\infty} \frac{1}{n}\int_{\{-2n< v\le -n\}} a(t,x,u,\nabla u)\cdot\nabla v\psi\,dx\,dt = \int_Q \psi d\mu_s^-, \end{aligned} \tag{4.1.8}$$

REMARK 4.2. Notice that, if u is a renormalized solution of (4.1.1), then

$$\begin{aligned} &(S(u-g))_t - \operatorname{div}(a(t,x,u,\nabla u)S'(u-g)) + S''(u-g)a(t,x,u,\nabla u)\cdot\nabla(u-g) \\ &= S'(u-g)f + S''(u-g)G\cdot\nabla(u-g) - \operatorname{div}(GS'(u-g)) \end{aligned} \tag{4.1.9}$$

is satisfied in the sense of distributions. Hence we can put as test functions not only functions in $C_0^\infty(Q)$ but also in $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

4.2. Statement of results and intermediary lemmas

In what follows the variable ϵ will belong to a sequence of positive numbers converging to zero. Let $a_\epsilon : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of functions satisfying the hypothesis $H(c_0, c_1, c_2, b_0)$. Assume that there exists a function $a_0 : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the hypothesis $H(c_0, c_1, c_2, b_0)$, and such that

$$\lim_{\epsilon\to 0} a_\epsilon(t,x,s_\epsilon,\zeta_\epsilon) = a_0(t,x,s,\zeta), \tag{4.2.1}$$

for every sequence $(s_\epsilon, \zeta_\epsilon) \in \mathbb{R}\times\mathbb{R}^N$ which converges to (s,ζ) and for almost $(t,x) \in Q$. Fixed $\mu \in \mathcal{M}_b(Q)$, we consider a special type of approximating sequence μ_ϵ, defined as follows.

DEFINITION 4.3. Let $\mu \in \mathcal{M}_b(Q)$ be decomposed as $\mu = f + F + g_t + \mu_s^+ - \mu_s^-$, with $f \in L^1(Q)$, $F = -\operatorname{div}(G)$, $G \in (L^{p'}(Q))^N$ and $g_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$. Let (μ_ϵ) be a sequence of measures in $\mathcal{M}_b(Q)$, we say that (μ_ϵ) has a splitting $(f_\epsilon, F_\epsilon, g_\epsilon^t, \lambda_\epsilon^\oplus, \lambda_\epsilon^\ominus)$ converging to μ. If for every ϵ the measure μ_ϵ can be decomposed as

$$\mu_\epsilon = f_\epsilon + F_\epsilon + g_\epsilon^t + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus, \tag{4.2.2}$$

and the following holds

(i) (f_ϵ) is a sequence of $C_c^\infty(Q)$ functions converging to f weakly in $L^1(Q)$;
(ii) (G_ϵ) is a sequence of functions in $(C_c^\infty(Q))^N$ that converges to g strongly in $(L^{p'}(Q))^N$;
(iii) (g_ϵ^t) is a sequence of functions in $C_c^\infty(Q)$ that converges to g_t in $L^p(0,T;V)$;
(iv) $(\lambda_\epsilon^\oplus)$ is a sequence of non-negative measures in $\mathcal{M}_b(Q)$ such that $\lambda_\epsilon^\oplus = \lambda_{\epsilon,0}^{1,\oplus} - \operatorname{div}(\lambda_{\epsilon,0}^{2,\oplus}) + \lambda_{\epsilon,s}^\oplus$ with $(\lambda_{\epsilon,0}^{1,\oplus} \in L^1(Q)$, $\lambda_{\epsilon,0}^{2,\oplus} \in (L^{p'}(Q))^N$ and $\lambda_{\epsilon,s}^\oplus \in \mathcal{M}_s^+(Q))$ that converges to μ_s^+ in the narrow topology of measures;
(v) $(\lambda_\epsilon^\ominus)$ is a sequence of non-negative measures in $\mathcal{M}_b(Q)$ such that $\lambda_\epsilon^\ominus = \lambda_{\epsilon,0}^{1,\ominus} - \operatorname{div}(\lambda_{\epsilon,0}^{2,\ominus}) + \lambda_{\epsilon,s}^\ominus$ with $(\lambda_{\epsilon,0}^{1,\ominus} \in L^1(Q)$, $\lambda_{\epsilon,0}^{2,\ominus} \in (L^{p'}(Q))^N$ and $\lambda_{\epsilon,s}^\ominus \in \mathcal{M}_s^+(Q))$ that converges to μ_s^- in the narrow topology of measures.

Moreover, let $u_0^\epsilon \in C_0^\infty(\Omega)$ that approaches u_0 in $L^1(\Omega)$, notice that this approximation can be easily obtained via a standard convolution arguments and we can also assume

$$\|\mu_\epsilon\|_{L^1(Q)} \le C|\mu|; \quad \|u_{0,\epsilon}\|_{L^1(\Omega)} \le C\|u_0\|_{L^1(\Omega)}.$$

Remark 4.4. Let us introduce the following function that we will often use in the following

$$H_n(s) = \chi_{[-n,n]}(s) + \frac{2n-|s|}{n}\chi_{\{n<|s|\leq 2n\}}(s), \quad \overline{H}_n(s) = \int_0^s H_n(\tau)d\tau,$$

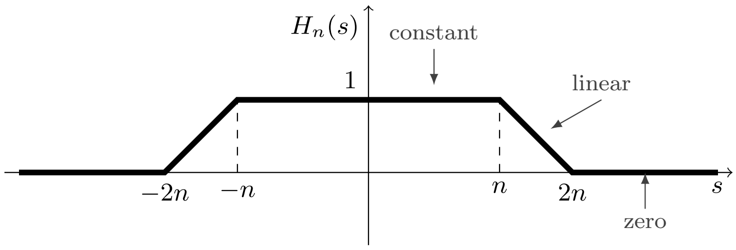


Figure 14. The function $H_n(s)$

and another auxiliary function introduced in terms of $H_n(s)$

$$B_n(s) = 1 - H_n(s).$$

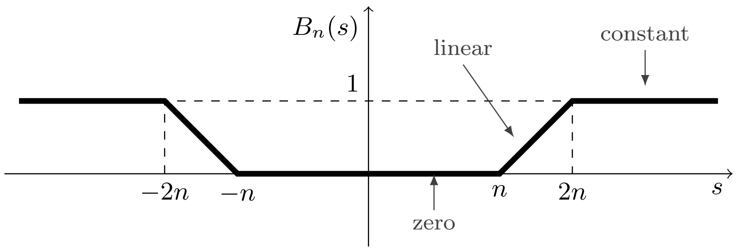


Figure 15. The function $B_n(s)$

Proposition 4.5. *Let $v = u - g$ be a renormalized solution of problem* (4.1.1). *Then, for every, $k > 0$, we have*

$$\int_Q |\nabla T_k(v)|^p dx\, dt \leq C(k+1),$$

where C is a positive constant not depending on k.

For a proof of the above proposition see [**Pe1**, Proposition 2].

Remark 4.6. If we decompose the measures, μ_ϵ, $\lambda^{\oplus}_\epsilon$, $\lambda^{\ominus}_\epsilon$ respectively as $\mu_\epsilon = \mu_{\epsilon,0} + \mu_{\epsilon,s}$, $\lambda^{\oplus}_\epsilon = \lambda^{\oplus}_{\epsilon,0} + \lambda^{\oplus}_{\epsilon,s}$ $(\lambda^{\oplus}_{\epsilon,0} = \lambda^{1,\oplus}_{\epsilon,0} - \mathrm{div}(\lambda^{2,\oplus}_{\epsilon,0}))$, $\lambda^{\ominus}_\epsilon = \lambda^{\ominus}_{\epsilon,0} + \lambda^{\ominus}_{\epsilon,s}$ $(\lambda^{\ominus}_{\epsilon,0} = \lambda^{1,\ominus}_{\epsilon,0} - \mathrm{div}(\lambda^{2,\ominus}_{\epsilon,0}))$, with $\mu_{\epsilon,0}, \lambda^{\oplus}_{\epsilon,0}$, $\lambda^{\ominus}_{\epsilon,0}$ in $\mathcal{M}_0(Q)$, and $\mu_{\epsilon,s}$, $\lambda^{\oplus}_{\epsilon,s}$, $\lambda^{\ominus}_{\epsilon,s}$ in $\mathcal{M}_s(Q)$, then clearly $\lambda^{\oplus}_{\epsilon,0}$, $\lambda^{\ominus}_{\epsilon,0}$, $\lambda^{\oplus}_{\epsilon,s}$, $\lambda^{\ominus}_{\epsilon,s}$ are non-negative, $\mu_{\epsilon,0} = f_\epsilon + F_\epsilon + g_\epsilon + \lambda^{\oplus}_{\epsilon,0} - \lambda^{\ominus}_{\epsilon,0}$ and $\mu_{\epsilon,s} = \lambda^{\oplus}_{\epsilon,s} - \lambda^{\ominus}_{\epsilon,s}$. In particular we have

$$0 \leq \mu^{+}_{\epsilon,s} \leq \lambda^{\oplus}_{\epsilon,s}, \quad 0 \leq \mu^{-}_{\epsilon,s} \leq \lambda^{\ominus}_{\epsilon,s}. \tag{4.2.3}$$

We are interested in the asymptotic behaviour of a sequence of renormalized solutions (u_ϵ) to the problem

$$\begin{cases} (u_\epsilon)_t - \mathrm{div}(a(t,x,u_\epsilon,\nabla u_\epsilon)) = \mu_\epsilon & \text{in } Q := (0,T)\times\Omega, \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_\epsilon(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.2.4}$$

in the sense of Definition 4.1. Our main result reads as follows.

THEOREM 4.7. *Let $(a_\epsilon), a_0$ be functions satisfying $H(c_0, c_1, c_2, b_0)$ and (4.2.1). Let $\mu \in \mathcal{M}_b(Q)$ be decomposed as $f + F + g_t + \mu_s^+ - \mu_s^-$, and let (μ_ϵ) a sequence of measures in $\mathcal{M}_b(Q)$ which have a splitting $(f_\epsilon, F_\epsilon, g_\epsilon, \lambda_\epsilon^\oplus, \lambda_\epsilon^\ominus)$ converging to μ. Assume that u_ϵ is a renormalized solution of (4.2.4). Then there exists a subsequence, still denoted by (u_ϵ), and a renormalized solution u to the problem*

$$\begin{cases} u_t - \operatorname{div}(a_0(t,x,u,\nabla u)) = \mu & \text{in } Q := (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.2.5}$$

such that (u_ϵ) converges to u a.e. in Q, and $(v_\epsilon) = (u_\epsilon - g_\epsilon)$ converges to $v = u - g$ a.e. in Q.

REMARK 4.8. The convergence of u_ϵ to u is not merely pointwise. The kind of convergences obtained are listed in Proposition 4.16, where the existence of the limit function u is obtained.

REMARK 4.9. Let z_ν be a sequence of functions such that

$$z_\nu \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \|z_\nu\|_{L^\infty(\Omega)} \le k,$$
$$z_\nu \to T_k(u_0) \text{ a.e. in } \Omega \text{ as } \nu \text{ tends to infinity},$$
$$\frac{1}{\nu}\|z_\nu\|^p_{W_0^{1,p}(\Omega)} \to 0 \text{ as } \nu \text{ tends to infinity.}$$

Then, for fixed $k > 0$, and $\nu > 0$, we denote by $(T_k(v))_\nu$ (Landes-time regularization of the truncate function $T_k(v)$ introduced in [**La**] and used in several articles (see [**BDGO, BP, DO2**]) the unique solution of the problem

$$\frac{dT_k(v)_\nu}{dt} = \nu(T_k(v) - T_k(v)_\nu) \quad \text{in the sense of distributions,}$$
$$T_k(v)_\nu = z_\nu \quad \text{in } \Omega,$$

therefore, $T_k(v)_\nu \in L^p(0,T; W_0^{1,p}(\Omega)\cap L^\infty(Q))$ and $\frac{dT_k(v)}{dt} \in L^p(0,T;W_0^{1,p}(\Omega))$, and it can be proved that, up to a subsequences, as ν diverges

$$T_k(v)_\nu \to T_k(v) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q,$$
$$\|T_k(v)_\nu\|_{L^\infty(Q)} \le k \quad \forall \nu > 0.$$

Then choosing this approximation in parabolic case with fact that (μ_ϵ) approximates μ in the sense of Definition 4.3. Hence we obtain, as consequence of the strong convergence of truncates the existence of renormalized solution of (4.2.5) obtained as stated in the following theorem.

THEOREM 4.10. *Let a_0 be a function satisfying $H(c_0, c_1, c_2, b_0)$ and $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_b(Q)$. Then there exists a renormalized solution u to the problem*

$$\begin{cases} u_t - \operatorname{div}(a_0(t,x,u,\nabla u)) = \mu & \text{in } Q := (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

We recall that a sequence (μ_ϵ) of non-negative measures converges to μ in the narrow topology if and only if $(\mu_\epsilon(Q))$ converges to $\mu(Q)$ and (1.12.8) holds for every $\varphi \in C_c^\infty(Q)$. In particular a sequence (μ_ϵ) of non-negative measures converges to μ in the narrow topology if and only if (1.12.8) holds for every $\varphi \in C_c(\overline{Q})$. The following lemma states a consequence result of the Dunford-Pettis theorem.

LEMMA 4.11. *Let (ρ_ϵ) be a sequence in $L^1(Q)$ converging to ρ weakly in $L^1(Q)$ and (σ_ϵ) a bounded sequence in $L^\infty(Q)$ converging to σ a.e. in Q. Then*

$$\lim_{\epsilon\to 0}\int_Q \rho_\epsilon\sigma_\epsilon dx\, dt = \int_Q \rho\sigma\, dx\, dt$$

Next, we need to localize some integrals near the support of $\mu_s \in \mathcal{M}_s(Q)$ (singular measure with respect to $p-$capacity). This will be done in terms of the following cut-off functions (see [**Pe1**, Lemma 5]).

LEMMA 4.12. *Let μ_s be a measure in $\mathcal{M}_s(Q)$, and let μ_s^+, μ_s^- be respectively the positive and the negative part of μ_s. Then for every $\delta > 0$, there exists two functions $\psi_\delta^+, \psi_\delta^-$ in $C_0^1(Q)$, such that the following hold*

(i) $0 \le \psi_\delta^+ \le 1$ *and* $0 \le \psi_\delta^- \le 1$ *on* Q,
(ii) $\lim_{\delta\to 0} \psi_\delta^+ = \lim_{\delta\to 0} \psi_\delta^- = 0$ *strongly in* $L^p(0,T;W_0^{1,p}(\Omega))$ *and weakly–* in* $L^\infty(Q)$,
(iii) $\lim_{\delta\to 0}(\psi_\delta^+)_t = \lim_{\delta\to 0}(\psi_\delta^-)_t = 0$ *strongly in* $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$,
(iv) $\int_Q \psi_\delta^- d\mu_s^+ \le \delta$ *and* $\int_Q \psi_\delta^+ d\mu_s^- \le \delta$,
(v) $\int_Q (1-\psi_\delta^+\psi_\eta^+) d\mu_s^+ \le \delta + \eta$ *and* $\int_Q (1-\psi_\delta^-\psi_\eta^-) d\mu_s^- \le \delta+\eta$ *for all* $\eta > 0$.

LEMMA 4.13. *Let μ_s be a measure in $\mathcal{M}_s(\Omega)$, decomposed as $\mu_s = \mu_s^+ - \mu_s^-$, with μ_s^+ and μ_s^- concentrated on two disjoint subsets E^+ and E^- of zero $p-$capacity. Then, for every $\delta > 0$, there exists two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+\backslash K_\delta^+) \le \delta, \quad \mu_s^-(E^-\backslash K_\delta^-) \le \delta, \tag{4.2.6}$$

and there exists ψ_δ^+, $\psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+, \psi_\delta^- \equiv 1 \quad \textit{respectively on } K_\delta^+, K_\delta^-, \tag{4.2.7}$$

$$0 \le \psi_\delta^+, \psi_\delta^- \le 1, \tag{4.2.8}$$

$$\mathrm{supp}(\psi_\delta^+) \cap \mathrm{supp}(\psi_\delta^-) \equiv \emptyset. \tag{4.2.9}$$

Moreover

$$\|\psi_\delta^+\|_S \le \delta, \quad \|\psi_\delta^-\|_S \le \delta, \tag{4.2.10}$$

and, in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \le \frac{\delta}{3}, \tag{4.2.11}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \le \frac{\delta}{3}, \tag{4.2.12}$$

and both ψ_δ^+ and ψ_δ^- converge to zero weakly– in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes. Moreover, if $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are as in* (4.2.2) *we have*

$$\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = \omega(\epsilon,\delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \le \delta, \tag{4.2.13}$$

$$\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = \omega(\epsilon,\delta), \quad \int_Q \psi_\delta^+ d\mu_s^- \le \delta, \tag{4.2.14}$$

$$\int_Q (1-\psi_\delta^+\psi_\eta^+) d\lambda_\epsilon^\oplus = \omega(\epsilon,\delta,\eta), \quad \int_Q (1-\psi_\delta^+\psi_\eta^+) d\mu_s^+ \le \delta+\eta, \tag{4.2.15}$$

$$\int_Q (1-\psi_\delta^-\psi_\eta^-) d\lambda_\epsilon^\ominus = \omega(\epsilon,\delta,\eta), \quad \int_Q (1-\psi_\delta^-\psi_\eta^-) d\mu_s^- \le \delta+\eta. \tag{4.2.16}$$

For a proof of the above lemma see [**Pe1**, Lemma 5].

REMARK 4.14. If $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ satisfy (iii) and (iv) of Definition 4.3, respectively, and ψ_δ^- and ψ_δ^+ are the functions defined in Lemma 4.12, as an easy consequence of the narrow convergence we obtain

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = 0, \quad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = 0, \tag{4.2.17}$$

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^+\psi_\eta^+) d\lambda_\epsilon^\oplus = 0, \quad \lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^-\psi_\eta^-) d\lambda_\epsilon^\ominus = 0. \tag{4.2.18}$$

4.3. Existence of a limit function

The following lemma is the main tool in order to establish the fundamental a priori estimates for the sequence (u_ϵ).

LEMMA 4.15. *Let u, v as defined before, and assume that there exists $C > 0$ such that*

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad \|v\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$
$$\int_Q |\nabla T_k(u)|^p dx\,dt \leq Ck, \quad \int_Q |\nabla T_k(v)|^p dx\,dt \leq C(k+1), \tag{4.3.1}$$

for every $k > 0$. Then there exists $C = C(N, M, p) > 0$ such that

(i) $\text{meas}\{|u| \geq k\} \leq Ck^{-(p-1+\frac{p}{N})}$, $\quad \text{meas}\{|v| \geq k\} \leq Ck^{-(p-1+\frac{p}{N})}$,
(ii) $\text{meas}\{|\nabla u| \geq k\} \leq Ck^{-(p-\frac{N}{N+1})}$, $\quad \text{meas}\{|\nabla v| \geq k\} \leq Ck^{-(p-\frac{N}{N+1})}$.

PROOF. (i) We can improve this kind of estimate by using a suitable Gagliardo-Nirenberg type inequality (see [**DiB**, Proposition 3.1]) which asserts that is $w \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, with $q \geq 1$, $\sigma \geq 1$. Then $w \in L^\sigma(Q)$ with $\sigma = q\frac{N+\rho}{N}$ and

$$\int_Q |w|^\sigma dx\,dt \leq C\|w\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla w|^q dx\,dt.$$

Indeed, in this way we obtain

$$\int_Q |T_k(u)|^{p+\frac{p}{N}} dx\,dt \leq Ck,$$

and so, we can write

$$K^{p+\frac{p}{N}} \text{meas}\{|u| \geq k\} \leq \int_{\{|u|\geq k\}} |T_k(u)|^{p+\frac{p}{N}}\, dx\,dt \leq \int_Q |T_k(u)|^{p+\frac{p}{N}}\, dx\,dt \leq Ck,$$

Then,

$$\text{meas}\{|u| \geq k\} \leq \frac{C}{k^{p-1+\frac{p}{N}}}.$$

(ii) We are interested about a similar estimate on the gradients of functions u; let us emphasize that these estimates hold true. First of all, observe that

$$\text{meas}\{|\nabla u| \neq \lambda\} \leq \text{meas}\{|\nabla u| \neq \lambda,\ |u| \leq k\} + \text{meas}\{|\nabla u| \neq \lambda,\ |u| > k\}$$

with regard to the first term in the right hand side, we have

$$\begin{aligned} \text{meas}\{|\nabla u| \neq \lambda,\ |u| \leq k\} &\leq \frac{1}{\lambda^p} \int_{\{|\nabla u|\geq\lambda;|u|\leq k\}} |\nabla u|^p dx \\ = \frac{1}{\lambda^p} \int_{\{|u|\leq k\}} |\nabla u|^p dx &= \frac{1}{\lambda^p} \int_Q |\nabla T_k(u)|^p dx \leq \frac{Ck}{\lambda^p}, \end{aligned} \tag{4.3.2}$$

while for the last term, thanks to (i), we can write

$$\text{meas}\{|\nabla u| \geq \lambda,\ |u| > k\} \leq \text{meas}\{|u| \geq k\} \leq \frac{\overline{C}}{K^\sigma},$$

with $\sigma = p - 1 + \frac{p}{N}$. So, finally, we obtain

$$\text{meas}\{|\nabla u| \geq \lambda\} \leq \frac{\overline{C}}{k^\sigma} + \frac{Ck}{\lambda^p},$$

and we obtain a better estimate by taking the minimum over k of the right-hand side; the minimum is achieved for the value

$$k_0 = \big(\frac{\sigma C}{\overline{C}}\big)^{\frac{1}{\sigma+1}} \lambda^{\frac{p}{\sigma+1}}$$

and so we obtain the desired estimate

$$\text{meas}\{|\nabla u| \geq \lambda\} \leq C\lambda^{-\gamma}$$

with $\gamma = p(\frac{\sigma}{\sigma+1}) = \frac{Np+p-N}{N+1} = p - \frac{N}{N+1}$. Then, we found that u (resp v) is uniformly bounded in the Marcinkiewicz space $\mathcal{M}^{p-1+\frac{p}{N}}(Q)$ and ∇u (resp ∇v) is equi-bounded in $\mathcal{M}^\gamma(Q)$, with $\gamma = p - \frac{N}{N+1}$. □

From now we always assume that (a_ϵ), a_0 are functions satisfying $H(c_0,c_1,c_2,b_0)$ and (4.2.1), that $\mu \in \mathcal{M}_b(Q)$ is decomposed as $f+F+g_t+\mu_s$, $f\in L^1(Q)$, $F\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_t\in L^p(0,T;V)$, $\mu_s\in\mathcal{M}_s(Q)$, and that (μ_s) is a sequence of measure in $\mathcal{M}_b(Q)$, which have a splitting $(f_\epsilon,F_\epsilon,g_\epsilon,\lambda_\epsilon^\oplus,\lambda_\epsilon^\ominus)$ converging to μ. We shall denotes by u_ϵ a renormalized solution of (4.2.4) with μ_ϵ as datum. Hence it satisfies

$$\begin{aligned}&\int_0^T \langle (v_\epsilon)_t,\varphi\rangle\,dt+\int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt\\&=\int_Q f_\epsilon\varphi\,dx\,dt+\int_0^T\langle F_\epsilon,\varphi\rangle\,dx\,dt+\int_Q\varphi\,d(\lambda_\epsilon^\oplus-\lambda_\epsilon^\ominus),\end{aligned}\tag{4.3.3}$$

for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$, $\varphi_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, with $\varphi(T,0)=0$.

As a first step, we find a function $u\in L^\infty(0,T;L^1(\Omega))$ such that $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$ which is the limit, up to a subsequence, of (u_ϵ) in suitable topology.

Proposition 4.16. *Let $\mu_\epsilon\in\mathcal{M}_b(Q)$, $(u_{0,\epsilon})\in L^1(\Omega)$, with $\sup_\epsilon|\mu_\epsilon(Q)|<\infty$ and $\|u_{0,\epsilon}\|_{1,\Omega}<\infty$. Let (u_ϵ) be a sequence of renormalized solutions of* (4.2.4)*, and let $v_\epsilon=u_\epsilon-g_\epsilon$. Then there exists $C>0$ such that*

$$\begin{aligned}\|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))}\le C,\quad &\int_Q|\nabla T_k(u_\epsilon)|^p dx\,dt\le Ck,\\ \|v_\epsilon\|_{L^\infty(0,T;L^1(\Omega))}\le C,\quad &\int_Q|\nabla T_k(v_\epsilon)|^p dx\,dt\le C(k+1),\end{aligned}\tag{4.3.4}$$

for every ϵ and for every $k>0$. Moreover there exists a subsequence, still denoted by u_ϵ (resp v_ϵ) and a measurable function u (resp v) such that the following convergence hold

(i) *u_ϵ (resp (v_ϵ)) converges to u (resp v) a.e. in Q;*
(ii) *u (resp v) belongs to $L^\infty(0,T;L^1(\Omega))$ and for every $k>0$, the sequence $(T_k(u_\epsilon))$ (resp $T_k(v_\epsilon)$) converges to $T_k(u)$ (resp $T_k(v)$) $\in L^p(0,T;W_0^{1,p}(\Omega))$ in the weak topology of $L^p(0,T;W_0^{1,p}(\Omega))$;*
(iii) *∇u_ϵ (resp (∇v_ϵ)) converges to ∇u (resp ∇v) a.e. in Q;*
(iv) *$a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ in the strong topology of the space $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<p-\frac{N}{N+1}$, while $a_\epsilon(t,x,u,\nabla T_k(u_\epsilon))$ converges to $a_0(t,x,u,\nabla T_k(u))$ in the weak topology of $(L^{p'}(Q))^N$ for every $k>0$.*

Proof. *Step 1. a priori estimates.* Let us choose $T_k(u_\epsilon)$ as test function in (4.3.3) and we integrate in $]0,t[$ to obtain

$$\int_\Omega\Theta_k(u_\epsilon(t))\,dx+\int_0^t\int_\Omega a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(u_\epsilon)\,dx\,dt=\int_0^t\int_\Omega T_k(u_\epsilon)d\mu_\epsilon+\int_\Omega\Theta_k(u_{0,\epsilon})\,dx\tag{4.3.5}$$

using (4.2.1) and the fact that $\|u_{0,\epsilon}\|_{L^1(\Omega)}$ and $\|\mu_\epsilon\|_{L^1(Q)}$ are bounded:

$$\int_\Omega\Theta_k(u_\epsilon)(t)\,dx+\int_0^t\int_\Omega|\nabla T_k(u_\epsilon)|^p dx\,dt\le Ck$$

Since $\Theta_k(s)\ge0$ and $|\Theta_1(s)|\ge|s|-1$, we obtain

$$\int_\Omega|u_\epsilon(t)|\,dx+\int_0^t\int_\Omega|\nabla T_k(u_\epsilon)|^p dx\,dt\le C(k+1),\quad\forall k>0,\forall t\in[0,T].$$

Taking the supremum on $(0,T)$. As a consequence we obtain the estimate of u_ϵ in $L^\infty(0,T;L^1(\Omega))$

$$\|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))}\le C.$$

We repeat here the same argument to get the estimate on v_ϵ: let us choose $T_k(v_\epsilon)$ as test function in (4.3.3). By integration by parts (recall that g_ϵ has compact support in Q, so that $(v_\epsilon(0)=u_\epsilon(0)=u_{0,\epsilon})$) and using

(4.2.1)

$$
\begin{aligned}
&\int_\Omega \Theta(v_\epsilon)(t)\,dx + \alpha \int_0^t \int_\Omega |\nabla u_\epsilon|^p \chi_{\{|v_\epsilon \leq k|\}}\,dx\,ds \\
&\quad \leq \int_\Omega \Theta_k(u_{0,\epsilon})\,dx + \int_Q f_\epsilon T_k(v_\epsilon)\,dx\,dt + \int_0^t \int_\Omega G_\epsilon \cdot \nabla u_\epsilon \chi_{\{|v_\epsilon \leq k|\}} dxds \\
&\qquad - \int_0^t \int_\Omega G_\epsilon \cdot \nabla g_\epsilon \chi_{\{|v_\epsilon \leq k|\}} dxds + \int_0^t \int_\Omega a(s,x,u_\epsilon,\nabla u_\epsilon) \cdot \nabla g_\epsilon \chi_{\{|v_\epsilon| \leq k\}} dsds \\
&\qquad + \int_Q T_k(v_\epsilon) d\lambda_\epsilon^\oplus - \int_Q T_k(v_\epsilon) d\lambda_\epsilon^\ominus,
\end{aligned}
$$

thanks to (4.2.2) and young's inequality,

$$
\begin{aligned}
&\int_\Omega \Theta(v_\epsilon)(t)\,dx + \frac{\alpha}{2} \int_0^t \int_\Omega |\nabla u_\epsilon|^p \chi_{\{|v_\epsilon \leq k|\}}\,dx\,ds \\
&\leq \int_Q |f_\epsilon| dx\,dt + C \int_Q |G_\epsilon|^{p'} dx\,dt + C \int_Q |\nabla g_\epsilon|^p dx\,dt \\
&\quad + C \int_Q |b(t,x)|^{p'} dx\,dt + k \int_\Omega |u_{0,\epsilon}| dx + k \int_Q d\lambda_\epsilon^\oplus + k \int_Q d\lambda_\epsilon^\ominus.
\end{aligned}
$$

Using that G_ϵ is bounded in $L^{p'}(Q)$, g_ϵ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f_ϵ, $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are bounded in $L^1(Q)$ and $u_{0,\epsilon}$ is bounded in $L^1(\Omega)$, we have

$$
\int_\Omega \Theta_1(v_\epsilon)\,dx \leq C \quad \forall t \in [0,T].
$$

In this way the same estimate of u_ϵ follows for v_ϵ in $L^\infty(0,T;L^1(\Omega))$:

$$
\begin{gathered}
\|v_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\
\int_Q |\nabla u_\epsilon|^p \chi_{\{|v_\epsilon| \leq k\}} dx\,dt \leq C(k+1),
\end{gathered}
$$

which yields that $T_k(v_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k > 0$ (recall that g_ϵ itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). Then

$$
\int_Q |\nabla T_k(v_\epsilon)|^p dx\,dt \leq C(k+1).
$$

Step 2. Up to a subsequence, u_ϵ is a Cauchy sequence in measure. We are going to prove now that, up to subsequences, u_ϵ converges almost everywhere in Q towards a measurable function u. Lemma 4.15 gives the usual estimates for parabolic equation with measure data, that is to say u_ϵ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ and in $L^\infty(0,T;L^1(\Omega))$, for which we can deduce that

$$
\lim_{k \to +\infty} \text{meas}\{(t,x) \in Q : |u_\epsilon| > k\} = 0 \quad \text{uniformly with respect to } u.
$$

From (4.3.4) we have that $T_k(u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. Now, if we multiply the approximating equation by $\mathcal{T}_k'(v_\epsilon)$, where $\mathcal{T}_k(s)$ is a $C^2(\mathbb{R})$, nondecreasing function such that $\mathcal{T}_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\mathcal{T}_k(s) = k$ for $|s| > k$, we obtain

$$
\begin{aligned}
&(\mathcal{T}_k(v_\epsilon))_t - \text{div}(a(t,x,u_\epsilon,\nabla u_\epsilon)\mathcal{T}_k'(v_\epsilon)) + a(t,x,u_\epsilon,\nabla u_\epsilon) \cdot \nabla v_\epsilon \mathcal{T}_k''(v_\epsilon) \\
&= \mathcal{T}_k'(v_\epsilon) f_\epsilon + \mathcal{T}_k''(v_\epsilon) G_\epsilon \cdot \nabla v_\epsilon - \text{div}(G_\epsilon \mathcal{T}_k'(v_\epsilon)) + (\lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus)\mathcal{T}_k'(v_\epsilon)
\end{aligned}
$$

in the sense of distributions. This implies, thanks to the last equality and to the fact that $\mathcal{T}_k'$ has compact support, that $\mathcal{T}_k(v_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ while its time derivative $(\mathcal{T}_k(v_\epsilon))_t$ is bounded in $L^p(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, hence a classical compactness result **[Si]** allows us to conclude that $\mathcal{T}_k(v_\epsilon)$ is compact in $L^2(Q)$. Thus for a subsequence, it also converges in measure, and almost everywhere in Q. Since we have, for $\sigma > 0$,

$$\begin{aligned}\operatorname{meas}\{(t,x):|v_n-v_m|>\sigma\} \le \operatorname{meas}\{(t,x):|v_n|>\frac{k}{2}\}+\operatorname{meas}\{(t,x):|v_n|>\frac{k}{2}\}\\ +\operatorname{meas}\{(t,x):|\mathcal{T}_k(v_n)-\mathcal{T}_k(v_m)|>\sigma\},\end{aligned}$$

by (4.3.4) for every fixed $\epsilon > 0$ we can choose $\overline{k}$ large enough to have

$$\operatorname{meas}\{(t,x):|v_n-v_m|>\sigma\} \le \operatorname{meas}\{(t,x):|\mathcal{T}_k(v_n)-\mathcal{T}_{\overline{k}}(v_m)|>\sigma\}+\epsilon, \tag{4.3.6}$$

for all $n, m \in \mathbb{N}$. The fact that $\mathcal{T}_k(v_\epsilon)$ converges in measure for every $k > 0$ implies, using (4.1.7), that, up to subsequences, v_ϵ also converges in measure and almost everywhere in Q. In particular, we have found out that there exists a measurable function v in $L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ such that $T_k(v)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$, and for a subsequences, not relabeled,

$$T_k(v_\epsilon) \to T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \text{ strongly in } L^p(Q) \text{ and a.e. in } Q.$$

We deduce that

$$v_\epsilon \to v \text{ a.e. in } Q,$$

and since g_ϵ strongly converges to g in $L^p(0,T;W_0^{1,p}(\Omega))$, there exists a measurable function u such that

$$u_\epsilon \to u \text{ a.e. in } Q,$$

The estimate (4.3.4) also imply that $u \in L^\infty(0,T;L^1(\Omega))$. Indeed, using Fatou's Lemma on the first term of the left-hand of

$$\int_\Omega |u_\epsilon(t)|\,dx + \int_0^t\int_\Omega |\nabla T_k(u_\epsilon)|^p dx\,dt \le C(k+1), \quad \forall k>0, \forall t \in [0,T].$$

where

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega))$$

and in addition

$$\int_Q |\nabla T_k(u)|^p dx\,dt \le Ck, \quad \int_Q |\nabla T_k(v)|^p dx\,dt \le C(k+1), \tag{4.3.7}$$

that is property (ii) holds.

Step 3. ∇u_ϵ is a Cauchy sequence in measure. Let us show that ∇u_ϵ is a Cauchy sequence in measure, which will yields $\nabla u_\epsilon \to \nabla u$ almost everywhere, for a convenient subsequence. Given $\delta > 0$ for every $\eta > 0$ and $k > 0$ one has

$$\begin{aligned}\{(t,x),|\nabla u_n-\nabla u_m|\ge\delta\} &\subseteq \{(t,x),|u_n|>k\}\cup\{(t,x),|u_m|>k\}\\ &\cup\{(t,x),|\nabla u_n|>k\}\cup\{(t,x),|\nabla u_m|>k\}\cup\{(t,x),|u_n-u_m|>\eta\}\\ &\cup\big\{(t,x),|\nabla u_n-\nabla u_m|\ge\delta,|u_n\le k|,|\nabla u_n|\le k,\\ &\quad |u_n|\le k,|\nabla u_m|\le k,|u_n-u_m|\le\eta\big\}.\end{aligned} \tag{4.3.8}$$

We will denote A_1 to A_6 the six sets of the right hand side. One could remark, in the sequel of the proof, that only the upper bound of the measure of A_6 uses the equation of which u_n and u_m are solutions. The other bounds use the boundedness of (u_n) and (∇u_n). Let us bound meas(A_1) and meas(A_2), we have

$$k\operatorname{meas}(A_1) \le \int_{A_1} |\nabla u_n| dx\,dt \le \int_0^T\int_\Omega |\nabla u_n|\,dx\,dt$$

hence

$$\operatorname{meas}(A_1) \le \frac{1}{k}\int_0^T\int_\Omega |\nabla u_n| dx\,dt \le \frac{C}{k} \le \varepsilon,$$

for k large enough, because (∇u_n) is bounded in $L^q((0,T)\times\Omega)$ for $q < p - \frac{N}{N+1}$ and hence in $L^1((0,T)\times\Omega)$. Let us fix k such that

$$\operatorname{meas}(A_1) \le \varepsilon, \quad \operatorname{meas}(A_2) \le \varepsilon \quad \forall n, m \in \mathbb{N},$$

Now let us bound meas(A_3), we have (u_n) is a Cauchy sequence in $L^1((0,T)\times\Omega)$ hence for a given n, there exist n_0 such that for $n,m\geq n_0$ one has

$$\text{meas}(A_3)\leq\varepsilon,$$

it is now sufficient to bound meas(A_4), and to choose η. Thanks to the monotonicity of A, we have $[a(t,x,s,\zeta_1)-a(t,x,s,\zeta_2)](\zeta_1-\zeta_2)>0$ for $\zeta_1-\zeta_2\neq 0$. Since the set of (ζ_1,ζ_2) such that: $\{(t,x),\ |s|\leq k,\ |\zeta_1|\leq k,\ |\zeta_2|\leq k$ and $|\zeta_1-\zeta_2|\geq\delta\}$ is compact and a is continuous with respect to ζ for almost all t and x, $[a(t,x,s,\zeta_1)-(a(t,x,s,\zeta_2)](\zeta_1-\zeta_2)$ reaches on this compact its minimum that we will denotes $\gamma(t,x)$, and that verifies $\gamma(t,x)>0$ a.e. Since $\gamma(t,x)>0$ a.e., there exists $\epsilon'>0$ such that, for all measurable set $A\subset(0,T)\times\Omega$,

$$\int_A\gamma\leq\varepsilon'\Longrightarrow\text{meas}(A)\leq\varepsilon,$$

hence, to obtain meas$(A_4)\leq\varepsilon$, it is sufficient to show that

$$\int_{A_4}\gamma\leq\varepsilon'. \tag{4.3.9}$$

By definition of γ and A_4, we have

$$\int_{A_4}\gamma\leq\int_{A_4}(a(t,x,u_n,\nabla u_m)-a(t,x,u_m,\nabla u_m))\cdot(\nabla u_n-\nabla u_m)\chi_{\{|u_n-u_m|\leq\eta\}}.$$

Moreover the term to be integrated is non-negative and $\nabla T_\eta(u_n-u_m)=(\nabla u_n-\nabla u_m)\chi_{\{|u_n-u_m|\leq\eta\}}$, hence we have

$$\int_{A_4}\gamma\leq\int_0^T(a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\cdot\nabla T_\eta(u_n-u_m),$$

if one chooses $\varphi=T_\eta(u_n-u_m)\in L^p(0,T;W^{1,p}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$, which satisfies $T_\eta(u_n-u_m)_t\in L^{p'}((0,T);W^{-1,p'}(\Omega))$, in equation in the sense of distributions written successively with u_n and u_m one gets

$$\begin{aligned}\int_0^T\langle(u_n-u_m)_t,T_\eta(u_n-u_m)\rangle+\int_0^T\int_\Omega(a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\cdot\nabla T_\eta(u_n-u_m)\\=\int_0^T\int_\Omega(\mu_n-\mu_m)T_\eta(u_n-u_m),\end{aligned}$$

that is (using Θ_η the primitive of T_η)

$$\begin{aligned}\int_\Omega\Theta_\eta(u_n-u_m)(T)-\int_\Omega\Theta_\eta(u_n-u_m)(0)+\int_0^T\int_\Omega(a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\cdot\nabla T_\eta(u_n-u_m)\\=\int_0^T\int_\Omega(\mu_n-\mu_m)T_\eta(u_n-u_m).\end{aligned}$$

Since the first term is non-negative ($\Theta_\eta(x)\geq 0$), and $\Theta_\eta(x)\leq\eta|x|$ one has

$$\begin{aligned}&\int_0^T\int_\Omega(a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\cdot\nabla T_\eta(u_n-u_m)\\&\leq\eta\int_0^T\int_\Omega|\mu_n-\mu_m|+\eta\int_\Omega|u_0^n-u_0^m|\leq 2\eta\left(|\mu(Q)|+\|u_0\|_{1,\Omega}\right).\end{aligned}$$

Then for η small enough, one has $\int_{A_4}\gamma\leq\varepsilon'$ and thus meas$(A_4)\leq\varepsilon$ and therefore for all $n,m\geq n_0$ we have

$$\text{meas}(\{|(\nabla u_n-\nabla u_m)(x)|\geq\delta\})\leq 4\varepsilon,$$

thus, we obtain that ∇u_ϵ is a Cauchy sequence in measure. Passing to a subsequence, we assume that

$$\nabla u_\epsilon\to\nabla u\quad\text{almost everywhere in } Q.$$

Similarly, we obtain the convergence a.e of v_ϵ, this gives

$$\nabla v_\epsilon\to\nabla v\text{ almost everywhere in } Q,$$

that is property (iii) holds.

It remains to prove (iv). By (4.3.5), Lemma 4.15, and (4.1.3), $a(t,x,u_\epsilon,\nabla u_\epsilon)$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<p-\frac{N}{N+1}$. Moreover, by (4.2.1), (i) and (iii), $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ a.e. in Q.

Hence by Vitali's Theorem, we have that $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ in the strong topology of $L^q(0,T;W_0^{1,q}(\Omega))$, $1\le q<p-\frac{N}{N+1}$. Finally, by (ii) and (4.1.3), the sequence $(a_\epsilon(t,x,u_\epsilon,\nabla T_k(u_\epsilon))$ is bounded in $L^{p'}(Q)$, which easily implies that it converges to $a_0(t,x,u,\nabla T_k(u))$ in the weak topology of $L^{p'}(Q)$. □

4.4. Proof of the main result

At this point we have a subsequence (u_ϵ) of renormalized solutions to (4.2.4) and a measurable function u with $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$ such that all the convergences stated in Proposition 4.16 hold. We have to prove that the function u is a renormalized solution to (4.2.5). By Proposition 4.16 (ii) condition (a) of Definition 4.1 is satisfied, while by (4.3.7) and Lemma 4.15, we obtain that u satisfies condition (4.1.6) of Definition 4.1. Hence, it is enough to prove (4.1.7). Let $S\in W^{2,\infty}(\mathbb{R})$, and let $\varphi\in C_0^1([0,T]\times\Omega)$. We choose $S'(v_\epsilon)\varphi$ as test function in the equation solved by u_ϵ, obtaining

$$\begin{aligned}&-\int_\Omega S(u_{0,\epsilon})\varphi(0)\,dx-\int_0^T\langle\varphi_t,S(v_\epsilon)\rangle+\int_Q S'(v_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt\\&+\int_Q S''(v_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla v_\epsilon)\cdot\nabla v_\epsilon\varphi\,dx\,dt=\int_Q S'(v_\epsilon)\varphi d\hat\mu_\epsilon+\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\oplus-\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\ominus.\end{aligned}\tag{4.4.1}$$

As $\mathrm{supp}(S')\subset[-M,M]$, we have

$$\int_Q a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon S''(v_\epsilon)\varphi\,dx\,dt=\int_Q a_\epsilon(t,x,u_\epsilon,\nabla T_M(v_\epsilon)\varphi)\,dx\,dt$$

To pass to the limit in this term, we need the following improvement of Proposition 4.16 (ii).

PROPOSITION 4.17. *Let $(a_\epsilon),a_0$ be functions satisfying $H(c_0,c_1,c_2,b_0)$ and (4.2.1). Let $\mu\in\mathcal{M}_b(Q)$ be fixed, and $\mu=f+F+g_t+\mu_s$, $f\in L^1(Q)$, $F\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $\mu_s\in\mathcal{M}_s(Q)$. Assume that (μ_ϵ) is a sequence of measures in $\mathcal{M}_b(Q)$ having a splitting $(f_\epsilon,F_\epsilon,g_{t,\epsilon},\lambda_\epsilon^\oplus,\lambda_\epsilon^\ominus)$ which converges to μ. Let (u_ϵ) a sequence of renormalized solutions of (4.2.4), and let u be its limit in the sense of Proposition 4.16. Then for every $k>0$ the sequence $(T_k(u_\epsilon))$ converges strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ to $T_k(u)$ as ϵ goes to 0.*

PROOF. It is sufficient to follow the lines of the long and not easy proof of the same result, for a fixed operator independent of u, for the elliptic case in [**DMOP**, Sections 5–8], for the parabolic case in [**Pe1**, Section 7]. The assumptions on a_ϵ allow to obtain some estimates for varying operators explicitly depending on u.

For any $\delta,\eta>0$, let $\psi_\delta^+,\psi_\eta^+,\psi_\delta^-$ and ψ_η^- as in Lemma 4.13 and let E^+ and E^- be the sets where, respectively, μ_s^+,μ_s^- are concentrated; setting

$$\Phi_{\delta,\eta}=\psi_\delta^+\psi_\eta^++\psi_\delta^-\psi_\eta^-.$$

Suppose that, the estimate near E,

$$I_1=\int_{\{|v_\epsilon|\le k\}}\Phi_{\delta,\eta}a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon-T_k(v)_\nu)\le\omega(\epsilon,\nu,\delta,\eta),\tag{4.4.2}$$

and far from E,

$$I_2=\int_{\{|v_\epsilon|\le k\}}(1-\Phi_{\delta,\eta})a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon-T_k(v)_\nu)\le\omega(\epsilon,\nu,\delta,\eta).\tag{4.4.3}$$

Putting these statements together we obtain

$$\limsup_{\nu\to0,\epsilon\to0}\int_{\{|v_\epsilon|\le k\}}a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon-T_k(v)_\nu)\le0,\tag{4.4.4}$$

so that using the convergence of $(T_k(v)_\nu)$ to $T_k(v)$ in X we deduce

$$\limsup_{\epsilon\to0}\int_{\{|v_\epsilon|\le k\}}a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon-T_k(v))\le0,\tag{4.4.5}$$

since by the weak convergence of $T_k(v_\epsilon)$ to $T_k(v)$ in X, Proposition 4.16 implies that

$$\int_{\{|v_\epsilon|\le k\}}a(t,x,u,\nabla(T_k(v)+g_\epsilon))\cdot\nabla(T_k(v_\epsilon)-T_k(v))=\omega(\epsilon).\tag{4.4.6}$$

Then we obtain

$$\int_{\{|v_\epsilon|\le k\}} (a(t,x,u_\epsilon,\nabla u_\epsilon) - a(t,x,u,\nabla(T_k(v)+g_\epsilon))) \cdot \nabla(u_\epsilon - (T_k(v)+g_\epsilon)) = \omega(\epsilon),$$

we also have, using the convergence of ∇u_ϵ to ∇u a.e. in Q

$$(a(t,x,u_\epsilon,\nabla u_\epsilon)) \rightharpoonup a(t,x,u,\nabla u) \quad \text{in } (L^{p'}(Q))^N, \tag{4.4.7}$$

then we obtain

$$\limsup_{\epsilon\to 0} \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla T_k(v_\epsilon) \le \int_Q a(t,x,u,\nabla u)\cdot \nabla T_k(v).$$

so that by Proposition 4.16, since $(a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon+g_\epsilon))$ converges weakly in $(L^{p'}(Q))^N$ to some F_k, it follows that $F_k = a(t,x,u,\nabla(T_k(u)+g))$. We get

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \int_Q a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot \nabla(T_k(v_\epsilon)+g_\epsilon)\\
&\le \limsup_{\epsilon\to 0} \int_Q a(t,x,u_\epsilon,\nabla v_\epsilon)\cdot \nabla T_k(v_\epsilon) + \limsup_{\epsilon\to 0} \int_Q a(t,x,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot \nabla g_\epsilon\\
&\le \int_Q a(t,x,u,\nabla(T_k(v)+h))\cdot \nabla(T_k(v)+g).
\end{aligned}$$

We finally deduce

$$(T_k(v_\epsilon)) \quad \text{converges to } T_k(v) \text{ strongly in } X \text{ for all } k>0. \tag{4.4.8}$$

□

The next Lemma is devoted to establish the preliminary essential estimate.

LEMMA 4.18. *Near E we have the estimate*

$$I_1 = \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla(v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon,\nu,\delta,\eta).$$

PROOF. We have

$$I_1 = \int_Q \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla T_k(v_\epsilon) - \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla(T_k(v))_\nu.$$

so that, from Proposition 4.16 (iv) and since $a(t,x,u_\epsilon,\nabla T_k(v_\epsilon)+g_\epsilon)\cdot \nabla T_k(v)_\nu$ converges weakly in $L^1(Q)$ to $F_k\cdot\nabla(T_k(v))_\nu$, $\chi_{\{|v_\epsilon|\le k\}}$ converges to $\chi_{\{|v|\le k\}}$ a.e in Q, $\Phi_{\delta,\eta}$ converges to 0 a.e. in Q as $\delta\to 0$ and $\Phi_{\delta,\eta}$ takes its values in $[0,1]$, using Lemma 4.11, we have the first integral

$$\begin{aligned}
\int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla(T_k(v))_\nu &= \int_Q \chi_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot \nabla(T_k(v))_\nu\\
&= \int_Q \chi_{\{|v|\le k\}} \Phi_{\delta,\eta} F_k\cdot \nabla(T_k(v))_\nu + \omega(\epsilon)\\
&= \omega(\epsilon,\nu,\delta).
\end{aligned}$$

To obtain the second integral, We will use the function $k - T_k(s)$ (and its companion $k + T_k(s)$)

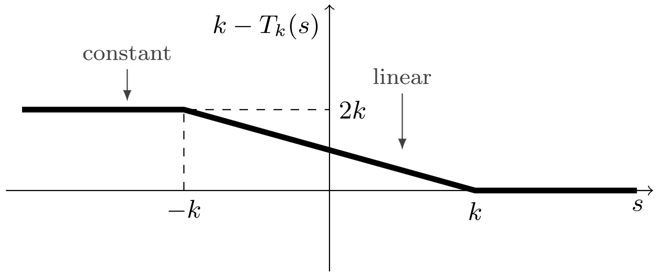


FIGURE 16. The function $k - T_k(s)$

we set, for any $n > k > 0$, and any $s \in \mathbb{R}$

$$\hat{S}_{n,k}(s) = \int_0^s (k - T_k(r))H_n(r)dr$$

where H_n is defined at Remark 4.4. We take $(S, \varphi) = (\hat{S}_{n,k}, \psi_\delta^+ \psi_\eta^+)$ as test function in (4.4.1), and we obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$A_1 = -\int_Q (\psi_\delta^+ \psi_\eta^+)_t \hat{S}_{n,k}(v_\epsilon)\, dx\, dt,$$

$$A_2 = \int_Q (k - T_k(v_\epsilon))H_n(v_\epsilon)a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(\psi_\delta^+ \psi_\eta^+)\, dx\, dt,$$

$$A_3 = -\int_Q \psi_\delta^+ \psi_\eta^+ a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(v_\epsilon)\, dx\, dt,$$

$$A_4 = \frac{2k}{n} \int_{\{-2n < v_\epsilon \le -n\}} \psi_\delta^+ \psi_\eta^+ a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla v_\epsilon\, dx\, dt,$$

$$A_5 = -\int_Q (k - T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+ \psi_\eta^+ d\hat{\mu}_{0,\epsilon},$$

$$A_6 = \int_Q (k - T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+ \psi_\eta^+ d(\lambda_\epsilon^\oplus + \lambda_\epsilon^\ominus).$$

Therefore, as in [**Pe1**], using the fact that $(\hat{S}_{n,k}(v_\epsilon))$ weakly converges to $\hat{S}_{n,k}(v)$ in X, $\hat{S}_{n,k}(v) \in L^\infty(Q)$ and (4.2.11) we obtain

$$A_1 = -\int_Q (\psi_\delta^+)_t \psi_\eta^+ \hat{S}_{n,k}(v) - \int_Q \psi_\delta^+ (\psi_\eta^+)_t \hat{S}_{n,k}(v) + \omega(\epsilon) = \omega(\epsilon, \delta).$$

Now since $v_\epsilon = T_{2n}(v_\epsilon)$ on $\mathrm{supp}(H_n(v_\epsilon))$ it follows from Proposition 4.16, (iv) that sequence $(a(t, x, u_\epsilon, \nabla(T_{2n}(v_\epsilon) + g_\epsilon))) \cdot \nabla(\psi_\delta^+ \psi_\eta^+)$ weakly converges to $F_{2n} \cdot \nabla(\psi_\delta^+ \psi_\eta^+)$ in $L^1(Q)$. From Lemma 4.11 and the convergence of $\psi_\delta^+ \psi_\eta^+$ in X to 0 as δ tends to 0, we obtain

$$A_2 = \int_Q (k - T_k(v_\epsilon))H_n(v_\epsilon)F_{2n} \cdot \nabla(\psi_\delta^+ \psi_\eta^+) + \omega(\epsilon) = \omega(\epsilon, \delta).$$

Because $0 \le \psi_\delta^+ \le 1$ (resp $0 \le \psi_\delta^- \le 1$). We then deduce

$$\begin{aligned} A_4 &= \frac{2k}{n} \int_{\{-2n < v_\epsilon \le -n\}} a(t, x, u_\epsilon, \nabla(T_{2n}(v_\epsilon) + g_\epsilon)) \cdot [\nabla(T_{2n}(v_\epsilon) + g_\epsilon) - \nabla g_\epsilon]\psi_\delta^+ \psi_\eta^+\, dx\, dt \\ &\le \frac{2k}{n} \int_{\{-2n < v_\epsilon \le -n\}} a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla v_\epsilon \psi_\eta^+\, dx\, dt + \omega(\epsilon, \delta, n). \end{aligned}$$

Therefore Lemma 4.12 implies

$$A_4 = \omega(\epsilon, \delta, n, \eta).$$

From the weak convergence of $((k-T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+\psi_\eta^+)$ to $(k-T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+$ in X and of the weak–* convergence of $(k-T_k(v_\epsilon))H_n(v_\epsilon)$ to $(k-T_k(v))H_n(v)$ in $L^\infty(Q)$ and a.e. in Q, the weak convergence of (f_ϵ) to f in $L^1(Q)$ and the strong convergence of (g_ϵ) to g in $(L^{p'}(Q))^N$. From Lemma 4.11 and the convergence of $\psi_\delta^+\psi_\eta^+$ to 0 in X and a.e. in Q as $\delta \to 0$

$$A_5 = \int_Q (k-T_k(v_\epsilon))H_n(v)\psi_\delta^+\psi_\eta^+ d\hat{\mu}_0 + \omega(\epsilon) = \omega(\epsilon,\delta).$$

We claim that the last term

$$A_6 \le 2k\int_Q \psi_\delta^+\psi_\eta^+ d(\lambda_\epsilon^\oplus + \lambda_\epsilon^\ominus) = 2k\int_Q \psi_\delta^+\psi_\eta^+ d(\mu_s^+ + \mu_s^-) + \omega(\epsilon).$$

Indeed, from Lemma 4.12 we have

$$A_6 \le \omega(\epsilon,\delta,\eta),$$

because A_3 does not depend on n. We then deduce from $\sum_{i=1}^6 A_i = 0$

$$A_3 = \int_Q \psi_\delta^+\psi_\eta^+ a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon) \le \omega(\epsilon,\delta,\eta).$$

Similarly, we take $(S,\varphi) = (\hat{S}_{n,k}, \psi_\delta^-\psi_\eta^-)$ as test function in (4.4.1), where $\hat{S}_{n,k}(s) = -\hat{S}_{n,k}(-s)$, we have, as before

$$\int_Q \psi_\delta^-\psi_\eta^- a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon) \le \omega(\epsilon,\delta,\eta).$$

So that using the two last inequalities we obtain

$$\int_Q \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon) \le \omega(\epsilon,\nu,\delta,\eta).$$

We finally deduce

$$I_1 = \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon,\nu,\delta,\eta).$$

□

REMARK 4.19. Note that: It is precisely for this estimate that we need the double cut functions $\psi_\delta^+\psi_\eta^+$. This results turns out to hold true even for more general functions ψ_η^+ and ψ_η^- in $W^{1,\infty}(Q)$, which satisfy

$$0 \le \psi_\eta^+ \le 1, \quad 0 \le \psi_\eta^- \le 1,$$
$$0 \le \int_Q \psi_\eta^+ d\mu_s^- \le \eta, \quad 0 \le \int_Q \psi_\eta^- d\mu_s^+ \le \eta.$$

LEMMA 4.20. *Far from E we have the estimate*

$$I_2 = \int_{\{|v_\epsilon|\le k\}} (1-\Phi_{\delta,\eta}) a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v_\epsilon) - T_k(v)_\nu).$$

PROOF. Now we follow the ideas in **[Pe1, Po1]**, for any $h > 2k > 0$, we define

$$w_\epsilon = T_{2k}(v_\epsilon - T_h(v_\epsilon) + T_k(v_\epsilon) - T_k(v)_\nu),$$

Note that $\nabla w_\epsilon = 0$ if $|v_\epsilon| > h + 4k$. As a consequence of the estimate on $T_k(v_\epsilon)$ in Proposition 4.16 we have w_ϵ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, we easily obtain

$$w_\epsilon \to T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu))$$

since $\|T_k(v)_\nu\|_{L^\infty(Q)} \le k$, we have also

$$\begin{cases} w_\epsilon = 2k \ \mathrm{sign}(v_\epsilon), \text{ in } \{|v_\epsilon| > h+2k\}, \quad |w_\epsilon| \le 4k, \quad w_\epsilon = w(\epsilon,\nu,h) \text{ a.e. in } Q, \\ \lim\limits_\epsilon w_\epsilon = T_{h+k}(v - (T_k(v))_\nu) - T_{h-k}(v - T_k(v)), \text{ a.e. in } Q \text{ and weakly in X.} \end{cases}$$

Let us take $w_\epsilon(1-\Phi_{\delta,\eta})$ as test functions in (4.3.3). We obtain

$$A_1 + A_2 + A_3 = A_4 + A_5,$$

where

$$A_1 = \int_0^T \langle v_{t,\epsilon}, w_\epsilon(1-\Phi_{\delta,\eta})\rangle\, dt,$$

$$A_2 = \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla w_\epsilon(1-\Phi_{\delta,\eta}),$$

$$A_3 = -\int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla \Phi_{\delta,\eta} w_\epsilon dx\, dt,$$

$$A_4 = w_\epsilon(1-\Phi_{\delta,\eta}) d\hat{\mu_0},$$

$$A_5 = \int_Q w_\epsilon(1-\Phi_{\delta,\eta}) d(\lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus).$$

Using the weak convergence of f_ϵ, again from the decomposition (4.2.2)

$$A_4 = \int_Q f_\epsilon w_\epsilon (1-\Phi_{\delta,\eta})\, dx\, dt + \int_Q G_\epsilon \cdot \nabla(w_\epsilon(1-\Phi_{\delta,\eta}))\, dx\, dt,$$

since f_ϵ converges to f weakly in $L^1(Q)$, from Lemma 4.11, we obtain

$$\int_Q f_\epsilon w_\epsilon(1-\Phi_{\delta,\eta})\, dx\, dt = \omega(\epsilon,\nu,h).$$

LEMMA 4.21. *Let $h,k>0$, and u_ϵ and $\Phi_{\delta,\eta}$ as before, then*

$$\int_{\{h\le |v_\epsilon|<h+k\}} |\nabla u_\epsilon|^p (1-\Phi_{\delta,\eta}) = \omega(\epsilon,h,\delta,\eta).$$

For a proof of the above lemma see [**Pe1**, Lemma 7].

Note that (g_ϵ) converges to g strongly in $(L^{p'}(Q))^N$, and $T_k(v)_\nu$ converges to $T_k(v)$ strongly in X. Then we deduce from Young's inequality and Lemma 4.21,

$$\begin{aligned}
&\int_Q G_\epsilon\cdot \nabla(w_\epsilon(1-\Phi_{\delta,\eta}))\, dx\, dt\\
&= \int_Q (1-\Phi_{\delta,\eta}) G\cdot \nabla(T_{h+k}(v-T_k(v)) - T_{h-k}(v-T_k(v)))\, dx\, dt + \omega(\epsilon,\nu)\\
&= \int_{\{h\le v<h+2k\}} (1-\Phi_{\delta,\eta}) G\cdot \nabla v\, dx\, dt + \omega(\epsilon,\nu,h)\\
&= \omega(h,\delta,\eta).
\end{aligned}$$

Then

$$A_4 = \omega(\epsilon,\nu,h,\delta,\eta).$$

To estimate of A_5, we have $|w_\epsilon| \le 2k$ and reasoning as in the proof of Lemma 4.21, and thanks to (4.2.13) – (4.2.16), we obtain

$$A_5 = \omega(\epsilon,\delta,\eta).$$

To estimate of A_1, we observe that, since $|T_k(v)_\nu| \le k$, w_ϵ can be written in the following way:

$$w_\epsilon = T_{h+k}(v_\epsilon - T_k(v)_\nu) - T_{h-k}(v_\epsilon - T_k(v_\epsilon)).$$

Hence, setting $G(t) = \int_0^t T_{h-k}(s-T_k(s))ds$, we have

$$\begin{aligned}
&\int_0^t \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta,\eta})\rangle\, dt\\
&= \int_0^t \langle (T_k(v)_\nu)_t, T_{h+k}(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\, dt\\
&\quad + \int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)_t (1-\Phi_{\delta,\eta})\ dx\, dt - \int_Q G(v_\epsilon)_t(1-\Phi_{\delta,\eta})\, dx\, dt
\end{aligned}$$

and since $|T_k(v)_\nu| \leq k$, using the definition of $T_k(v)_\nu$ we obtain

$$\begin{aligned}
&\int_0^t \langle (T_k(v)_\nu)_t, T_{h+k}(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\, dt \\
&= \nu \int_Q (T_k(v) - T_k(v)_\nu) T_{h+k}(v_\epsilon - T_k(v)_\nu)\, dx\, dt,
\end{aligned}$$

so that as ϵ tends to infinity, we have

$$\begin{aligned}
&\int_0^t \langle (T_k(v))_t, T_{h+k}(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\, dt \\
&= \omega(\epsilon) + \nu \int_Q (T_k(v) - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dx\, dt \\
&= \omega(\epsilon) + \nu \int_{\{|v|\leq k\}} (v - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dx\, dt \\
&\quad + \int_{\{v>k\}} (k - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dx\, dt \\
&\quad + \int_{\{v<-k\}} (-k - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\, dx\, dt.
\end{aligned}$$

since $|T_k(v)_\nu| \leq k$, last three terms are positives, hence we deduce by letting ϵ and ν to ∞,

$$\begin{aligned}
&\int_0^t \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta,\eta})\rangle\, dt \\
&= \omega(\epsilon) + \int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)_t (1-\Phi_{\delta,\eta})\, dx\, dt - \int_Q G(v_\epsilon)_t(1-\Phi_{\delta,\eta})\, dx\, dt \\
&= \omega(\epsilon) + \int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)\frac{\partial \Phi_{\delta\eta}}{dt}\, dx\, dt - \int_Q G(v_\epsilon)\frac{\partial \Phi_{\delta\eta}}{dt}\, dx\, dt \\
&\quad + \int_\Omega S_{h+k}(v_\epsilon - T_k(v)_\nu)(T)\, dx - \int_\Omega S_{h+k}(u_{0,\epsilon} - z_\nu)\, dx \\
&\quad - \int_\Omega G(v_\epsilon)(T)\, dx + \int_\Omega G(u_{0,\epsilon})\, dx.
\end{aligned}$$

Now we define the function $R(y) = S_{h+k}(y-z)\cdot G(y)$, with $|z| \leq k$. Then

$$\begin{cases}
R(y) = S_{h+k}(y+z) \geq 0, & |y| \leq k, \\
R'(y) = T_{h+k}(y-z) - T_{h-k}(y - T_k(y)) \geq 0, & y \geq k \geq z, \\
R'(y) \leq 0, & y \leq -k \leq z.
\end{cases}$$

Hence for every $z, |z| \leq k$, we have $R(y) \geq 0$ for every y in $\mathbb{R}$, we obtain

$$\int_\Omega S_{h+k}(v_\epsilon - T_k(v)_\nu)(T)\, dx - \int_\Omega G(v_\epsilon)(T)\, dx \geq 0,$$

letting ϵ and ν go to their limits,

$$\int_\Omega G(u_{u_{0,\epsilon}})\, dx - \int_\Omega S_{h+k}(u_{0,\epsilon} - z_\nu)\, dx = \int_\Omega G(u_0) - \int_\Omega S_{h+k}(u_0 - T_k(u_0)) + \omega(\epsilon, \nu),$$

Since we have $|G(u_0) - S_{h+k}(u_0 - T_k(u_0))| \leq 2k|u_0|\chi_{\{|u_0|>k\}}$, it follows that by letting h to $+\infty$,

$$\int_\Omega G(u_{0,\epsilon})\, dx - \int_\Omega S_{h+k}(u_{0,\epsilon} - z_\nu)\, dx = \omega(\epsilon, \nu, h)\,.$$

By the definition of $T_k(v)_\nu$,

$$\begin{aligned}
&\int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt - \int_Q G(v_\epsilon)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt \\
&= \int_Q (S_{h+k}(v - T_k(v) - G(v))\frac{d\Phi_{\delta\eta}}{dt}dx\,dt + \omega(\epsilon,\nu).
\end{aligned}$$

So, if $|v| \le h-k$, $S_{h+k}(v-T_k(v)) - G(v) = 0$, then $S_{h+k}(v-T_k(v)) - G(v)$ converges a.e. to 0 on Q, and since $v \in L^1(Q)$, by dominated convergence theorem

$$\int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt - \int_Q G(v_\epsilon)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt \ge \omega(\epsilon,\nu,h),$$

and so

$$\int_0^T \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta\eta})\rangle \ge \omega(\epsilon,\nu,h).$$

Now we estimate of A_2. Note that $\nabla w_\epsilon = 0$ if $|v_\epsilon| > h+4k$; then if we set $M = h+4k$, splitting the integral (A_2) on the sets $\{|v_\epsilon| > k\}$ and $\{|v_\epsilon| \le k\}$, using the fact that $T_h(v_\epsilon) = T_k(v_\epsilon) = v_\epsilon$ in $\{|v_\epsilon| \le k\}$ and $\nabla T_k(v_\epsilon)\chi_{|v_\epsilon|>k} = 0$. Then for $\{|v_\epsilon| \le M\}$ and $h \ge 2k$, we have

$$\begin{aligned}
A_2 &= \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla w_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla[(v_\epsilon - T_h(v_\epsilon)) - (T_k(v)_\nu)](1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>h\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla[(v_\epsilon - T_h(v_\epsilon))(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu - T_k(v)) + \nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{h<|v_\epsilon|>h+4k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu - T_k(v))(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt\,.
\end{aligned}$$

Using assumption (4.1.3), young's inequality, equi-integrability and Lemma 4.21, we see that for some $C = C(p,c_2)$,

$$\begin{aligned}
&\int_{\{h\le|v_\epsilon|<h+4k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt \\
&\le C\int_{\{h\le|v_\epsilon|<h+4k\}} (|\nabla u_\epsilon|^p + |\nabla g|^p + |b_0(t,x)|^{p'})(1-\Phi_{\delta\eta})\,dx\,dt \\
&\le \omega(\epsilon,h,\delta,\eta)\,.
\end{aligned}$$

Thanks to Proposition 4.16 and the fact that $T_k(v)_\nu$ converges strongly to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have

$$\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt=\omega(\epsilon),$$

$$\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu-T_k(v))(1-\Phi_{\delta\eta})\,dx\,dt=\omega(\epsilon,\nu).$$

Therefore,

$$A_2=\int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon-T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt+\omega(\epsilon,\nu,h,\delta,\eta).$$

□

Next we conclude the proof of Theorem 4.7.

LEMMA 4.22. *The function u is a renormalized solution of* (4.1.1).

PROOF. (i) Let $\varphi\in X\cap L^\infty(Q)$ such that $\varphi_t\in X'+L^1(Q)$, $\varphi(\cdot,T)=0$, and $S\in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on $\mathbb{R}$, $S(0)=0$. Let $M>0$ such that $\operatorname{supp}S'\subset[-M,M]$. Taking successively (φ,S), (φ,ψ_δ^+) and (φ,ψ_δ^-) as test functions in (4.4.1) applied to u_ϵ, we can write

$$A_1+A_2+A_3+A_4=A_5+A_6+A_7,$$
$$(A_2)_\delta^+ +(A_3)_\delta^+ +(A_4)_\delta^+ =(A_5)_\delta^+ +(A_6)_\delta^+ +(A_7)_\delta^+,$$
$$(A_2)_\delta^- +(A_3)_\delta^- +(A_4)_\delta^- =(A_5)_\delta^- +(A_6)_\delta^- +(A_7)_\delta^-$$

where

$$A_1=-\int_\Omega\varphi(0)S(u_{0,\epsilon})dt,\quad A_2=-\int_Q\varphi_tS(v_\epsilon)\,dx\,dt,$$
$$A_3=\int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt,$$
$$A_4=\int_Q S''(v_\epsilon)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon dx\,dt,$$
$$A_5=\int_Q S'(v_\epsilon)\varphi\hat\mu_\epsilon,\quad A_6=\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\oplus$$
$$A_7=-\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\ominus,$$

and

$$(A_2)_\delta^+=-\int_Q(\varphi\psi_\delta^+)_tS(v_\epsilon)\,dx\,dt,$$
$$(A_3)_\delta^+=\int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(\varphi\psi_\delta^+)\,dx\,dt,$$
$$(A_4)_\delta^+=\int_Q S''(v_\epsilon)\varphi\psi_\delta^+a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon dx\,dt,$$
$$(A_5)_\delta^+=\int_Q S'(v_\epsilon)\varphi\psi_\delta^+d\lambda_\epsilon^\oplus,$$
$$(A_6)_\delta^+=-\int_Q S'(v_\epsilon)\varphi\psi_\delta^+d\lambda_\epsilon^\ominus\,.$$

Since $(u_{0,\epsilon})$ converges to u_0 in $L^1(\Omega)$, and $(S(v_\epsilon))$ converges to $S(v)$, strongly in X, and weakly–* in $L^\infty(Q)$, it follows that

$$A_1=\int_\Omega\varphi(0)S(u_0)\,dx+\omega(\epsilon),\quad A_2=-\int_Q\varphi_tS(v)+\omega(\epsilon),$$
$$(A_2)_\delta^+=\omega(\epsilon,\delta),\quad(A_2)_\delta^-=\omega(\epsilon,\delta)\,.$$

Moreover, $T_M(v_\epsilon)$ converges to $T_M(v)$, then $T_M(v_\epsilon)+h_\epsilon$ converges to $T_k(v)+h$ strongly in X. Therefore,

$$\begin{aligned}A_3 &= \int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla(T_M(v_\epsilon)+h_\epsilon)\cdot\nabla\varphi\\ &= \omega(\epsilon)+\int_Q S'(v)a(t,x,u_\epsilon,\nabla(T_M(v)+h))\cdot\nabla\varphi\\ &= \omega(\epsilon)+\int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla\varphi,\end{aligned}$$

and

$$\begin{aligned}A_4 &= \int_Q S''(v_\epsilon)\varphi a(t,x,u_\epsilon,\nabla(T_M(v_\epsilon)+h_\epsilon))\cdot\nabla T_M(v_\epsilon)\\ &= \omega(\epsilon)+\int_Q S''(v)\varphi a(t,x,u,\nabla(T_M(v)+h))\cdot\nabla T_M(v)\\ &= \omega(\epsilon)+\int_Q S''(v)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,.\end{aligned}$$

In the same way, since $\psi_\delta^+,\psi_\delta^-$ converges to 0 in X,

$$\begin{aligned}(A_3)_\delta^+ &= \omega(\epsilon)+\int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla(\varphi\psi)_\delta^+ = \omega(\epsilon,\delta),\\ (A_3)_\delta^- &= \omega(\epsilon)+\int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla(\varphi\psi_\delta^-) = \omega(\epsilon,\delta),\\ (A_4)_\delta^+ &= \omega(\epsilon)+\int_Q S''(v)\varphi\psi_\delta^+ a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,\delta),\\ (A_4)_\delta^- &= \omega(\epsilon)+\int_Q S''(v)\varphi\psi_\delta^- a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,\delta),\end{aligned}$$

and (g_ϵ) strongly converges to g in $(L^{p'}(\Omega))^N$. Therefore,

$$\begin{aligned}(A_5) &= \int_Q S'(v_\epsilon)\varphi f_\epsilon + \int_Q S'(v_\epsilon)g_\epsilon\cdot\nabla\varphi + \int_Q S''(v_\epsilon)\varphi g_\epsilon\cdot\nabla T_M(v_\epsilon)\\ &= \omega(\epsilon)+\int_Q S'(v)\varphi f+\int_Q S'(v)g\cdot\nabla\varphi+\int_Q S''(v)\varphi g\cdot\nabla T_M(v)\\ &= \omega(\epsilon)+\int_Q S'(v)\varphi d\hat\mu_0\end{aligned}$$

Now, thanks to Proposition 4.16 and the properties of ψ_δ^+ and ψ_δ^-, we readily have

$$\begin{aligned}(A_5)_\delta^+ &= \omega(\epsilon)+\int_Q S'(v)\varphi\psi_\delta^+ d\hat\mu_\epsilon = \omega(\epsilon,\delta),\\ (A_5)_\delta^- &= \omega(\epsilon)+\int_Q S'(v)\varphi\psi_\delta^- d\hat\mu_\epsilon = \omega(\epsilon,\delta).\end{aligned}$$

Then

$$(A_6)_\delta^+ + (A_7)_\delta^+ = \omega(\epsilon,\delta),$$

and thanks to (4.2.14),

$$\begin{gathered}(A_7)_\delta^+ \le |\int_Q S'(v_\epsilon)\varphi\psi_\delta^+ d\lambda_\epsilon^\ominus| \le c\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = \omega(\epsilon,\delta),\\ (A_7)_\delta^- = \omega(\epsilon,\delta)\,.\end{gathered}$$

Then

$$(A_6)_\delta^+ = \int_Q S'(v_\epsilon)\varphi\psi_\delta^+ d\lambda_\epsilon^\oplus = \omega(\epsilon,\delta).$$

Moreover,

$$\begin{aligned}
A_6 &= \int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^{\ominus} \\
&= \int_Q S'(v_\epsilon)\varphi\psi_\delta^+ d\lambda_\epsilon^{\oplus} + \int_Q S'(v_\epsilon)\varphi(1-\psi_\delta^+)d\lambda_\epsilon^{\oplus} \\
&\le \omega(\epsilon,\delta) + \int_Q |S'(v_\epsilon)\varphi|(1-\psi_\delta^+)d\lambda_\epsilon^{\oplus} \\
&\le \omega(\epsilon,\delta) + \|S\|_{W^{2,\infty}(\mathbb{R})}\|\varphi\|_{L^\infty(Q)} \int_Q (1-\psi_\delta^+)d\lambda_\epsilon^{\oplus} \\
&\le \omega(\epsilon,\delta)\,.
\end{aligned}$$

Hence

$$A_6 = \omega(\epsilon) \text{ and } (A_7) = \omega(\epsilon).$$

Therefore, we finally obtain

$$\begin{aligned}
&-\int_\Omega \varphi(0)S(u_0)\,dx - \int_Q \varphi_t S(v) + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla\varphi \\
&+\int_Q S''(v)\varphi a(t,x,u,\nabla u)\cdot\nabla v \\
&= \int_Q S'(v)\varphi d\hat{\mu}_0
\end{aligned}$$

with $\varphi \in C_0^1([0,T]\times\Omega)$. By density argument we have (4.1.7) for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(\cdot,T) = 0$.

(ii) Next, we prove (4.1.8). We take $\varphi \in C_c^\infty(Q)$ and $(\varphi, S) = ((1-\psi_\delta -)\varphi, \overline{H}_n)$ as test functions in (4.1.7) and the same test functions in (4.4.1) applied to u_ϵ, we can write

$$\begin{aligned}
B_1^n + B_2^n &= B_3^n + B_4^n + B_5^n\,, \\
B_{1,\epsilon}^n + B_{2,\epsilon}^n &= B_{3,\epsilon}^n + B_{4,\epsilon}^n + B_{5,\epsilon}^n,
\end{aligned}$$

where

$$\begin{aligned}
B_1^n &= -\int_Q ((1-\psi_\delta^-)\varphi)_t \overline{H}_n(v)\,dx\,dt, \\
B_2^n &= \int_Q H_n(v)a(t,x,u,\nabla u)\cdot\nabla((1-\psi_\delta^-)\varphi)\,dx\,dt, \\
B_3^n &= \int_Q H_n(v)(1-\psi_\delta^-)\varphi d\hat{\mu}_0, \\
B_4^n &= \frac{1}{n}\int_{\{n<v\le 2n\}} (1-\psi_\delta^-)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt, \\
B_5^n &= -\frac{1}{n}\int_{\{-2n\le v<-n\}} (1-\psi_\delta^-)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt,
\end{aligned}$$

and

$$B^n_{1,\epsilon} = -\int_Q ((1-\psi^-_\delta)\varphi)_t \overline{H}_n(v_\epsilon)\,dx\,dt,$$
$$B^n_{2,\epsilon} = \int_Q H_n(v_\epsilon) a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla((1-\psi^-_\delta)\varphi)\,dx\,dt,$$
$$B^n_{3,\epsilon} = \int_Q H_n(v_\epsilon)(1-\psi^-_\delta)\varphi d(\hat{\mu}_{\epsilon,0} + \lambda^\oplus_\epsilon - \lambda^\ominus_\epsilon),$$
$$B^n_{4,\epsilon} = \frac{1}{n}\int_{\{n<v_\epsilon\le 2n\}} (1-\psi^-_\delta)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\,dx\,dt,$$
$$B^n_{5,\epsilon} = -\frac{1}{n}\int_{\{-2n\le v_\epsilon<-n\}} (1-\psi^-_\delta)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\,dx\,dt\,.$$

In the last terms, we go to the limit as $n\to+\infty$, since $(\overline{H}_n(v_\epsilon))$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation

$$B_{1,\epsilon} + B_{2,\epsilon} = B_{3,\epsilon} + B_\epsilon$$

where

$$B_{1,\epsilon} = -\int_Q ((1-\psi^-_\delta)\varphi)_t v_\epsilon,$$
$$B_{2,\epsilon} = \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla((1-\psi^-_\delta\varphi),$$
$$B_{3,\epsilon} = \int_Q (1-\psi^-_\delta)\varphi d\hat{\mu}_{\epsilon,0},$$
$$B_\epsilon = \int_Q (1-\psi^-_\delta)\varphi d(\lambda^\oplus_{\epsilon,0} - \lambda^\ominus_{\epsilon,0}) + \int_Q (1-\psi^-_\delta)\varphi d(\lambda^\oplus_{\epsilon,s} - \lambda^\ominus_{\epsilon,s})\,.$$

Clearly, $(B_{i,\epsilon}) - (B^n_i) = \omega(\epsilon,n)$ for $i=1,3$, from (4.2.14) – (4.2.16), we obtain

$$B^n_5 = \omega(\epsilon,n,\delta),$$
$$\frac{1}{n}\int_{\{n<v\le 2n\}} \psi^-_\delta\varphi a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,n,\delta)\,.$$

Thus

$$B^n_4 = \frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt + \omega(\epsilon,n,\delta)$$

since

$$|\int_Q (1-\psi^-_\delta)\varphi d\lambda^\ominus_\epsilon| \le \|\varphi\|_{L^\infty}\int_Q (1-\psi^-_\delta) d\lambda^\ominus_\epsilon,$$

it follows that $\int_Q (1-\psi^-_\delta)\varphi d\lambda^\ominus_\epsilon = \omega(\epsilon,n,\delta)$ from (4.2.16). And $|\int_Q \psi^-_\delta\varphi d\lambda^\oplus_\epsilon| \le \|\varphi\|_{L^\infty}\int_Q \psi^-_\delta\, d\lambda^\oplus_\epsilon$. Thus from (4.2.13) and (4.2.14), $\int_Q (1-\psi^-_\delta)\varphi d\lambda^\oplus_\epsilon = \int_Q \varphi d\mu^+_s + \omega(\epsilon,n,\delta)$. Then

$$B_\epsilon = \int_Q \varphi d\mu^+_s + \omega(\epsilon,n,\delta).$$

Therefore, by subtraction, we obtain successively

$$\frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt = \int_Q \varphi d\mu^+_s + \omega(\epsilon,n,\delta),$$
$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v = \int_\varphi d\mu^+_s,$$

which proves (4.1.8) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le v<2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le v<2n\}} \varphi\psi_\delta^+ a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&\quad + \lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+) a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \int_Q \varphi\psi_\delta^+ d\mu_s^+ + \lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+) a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \int_Q \varphi d\mu_s^+ + D
\end{aligned}$$

where

$$D = \int_Q \varphi(1-\psi_\delta^+)d\mu_s^+ + \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+)a(t,x,u,\nabla u)\nabla v\,dx\,dt = \omega(\epsilon).$$

Therefore, (4.1.8) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (4.1.8) by density, and similarly the second convergence. This complete the proof of Theorem 4.7. $\square$

CHAPTER 5

Standard porous medium problems with Leray–Lions operators and equi-diffuse measure

One of the recent advances in the investigation on nonlinear parabolic equations with a measure as forcing term is a paper by D. Blanchard, F. Petitta and H. Redwane [**BPR**] in which it has been introduced the notion of renormalized solutions to initial boundary value problems involving equations of the type

$$\begin{cases} b(u)_t - \operatorname{div}(a(t,x,\nabla u)) = \mu & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(u) = b(u_0) & \text{on } \{0\}\times\Omega, \end{cases} \tag{5.0.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, Q is the cylinder $(0,T)\times\Omega$, $(0,T)\times\partial\Omega$ being its lateral surface, $b : \mathbb{R} \mapsto \mathbb{R}$ is a C^1-increasing function, $b(u_0) \in L^2(\Omega)$ and μ is a Radon measure on Q. This setting contains as a particular case the doubly nonlinear diffusion equation, extending the standard porous media equation. The authors adapt to this setting the method of J. Droniou, A. Porretta and A. Prignet [**DPP**] dealing with the case $b = 1$ and diffuse measures with respect to the parabolic $p-$capacity. Recently, in [**PPP1, PPP2**], the authors proposed a new approach to the same problem ($b = 1$) and obtained the existence and uniqueness of solutions by approximation as a consequence of a stability result. This approach avoids to use the particular structure of the decomposition of the measure and it seems more flexible to handle a fairly general class of problems. In order to do that, they introduced a definition of renormalized solution which is closer to the one used for conservation laws used in [**BCW**] and to one of the existing formulations in the elliptic case [**DM, DMOP**]. Following the approach [**PPP2**], our goal is to to provide a new proof of this stability result, based on the properties of the truncations of renormalized solutions to the framework of the so-called standard porous medium equations of the type $v_t - \Delta_p \psi(v)$ with $\psi(v) = u$ and $\psi^{-1} = b$, ψ is a strictly increasing function. The approach, which does not need the strong convergence of the truncations of the solutions in the energy space, turns out to be easier and shorter than the original one. This Chapter is organized as follows. In Section 5.1, we give some preliminaries on diffuse measures and the fundamental capacitary estimate using parabolic $p-$capacity. The Section 5.2 is devoted to set the main assumptions and the new renormalized formulation of problem (5.0.1). In Section 5.3, we prove that this definition of renormalized solution does not depend on the classical decomposition of μ and it is equivalent to the basic formulation. In Section 5.4, we give the proof of the main result (Theorem 5.1) and we briefly sketch in Section 5.5 the proof of the uniqueness result.

5.1. Capacitary estimates and equi–diffuse measures

Diffuse measures play an important role in the study of boundary value problems with measures as source terms. Indeed, for such measures one expects to obtain counterparts, in some generalized framework, of existence and uniqueness results known in the variational setting. Properties of diffuse measures in connection with the resolution of nonlinear parabolic problems have been investigated in [**DPP**]. In that paper, the authors proved that for every $\mu \in \mathcal{M}_0(Q)$, there exists $f \in L^1(Q)$, $g \in L^p(0,T;V)$ and $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that

$$\mu = f + g + \chi \text{ in } \mathcal{D}'(Q). \tag{5.1.1}$$

Note that the decomposition in (5.1.1) is not uniquely determined and the presence of the term g (depends on t) is essentially due to the presence of diffuse measures which charges sections of the parabolic cylinder Q and gives some extra difficulties in the study of this type of problems; in particular the parabolic case with absorption term $h(u)$. The main reason is that a solution of

$$u_t - \Delta_p u + h(u) = \mu = f + \chi + g \text{ in } Q$$

is meant in the sense that $v = u - g$ satisfies

$$v_t - \Delta_p(v + g) + h(v + g) = f + \chi \text{ in } Q.$$

However, since no growth restriction is made on h, the proof is a hard technical issue if g is not bounded. For further considerations on this fact we refer to [**BP**] (see also [**BMP, PPP1**] and references therein). In [**PPP1**], the authors also proved the following approximation theorem for an arbitrary diffuse measure that is essentially independent on the decomposition of the measure data.

THEOREM 5.1. *Let $\mu \in \mathcal{M}_0(Q)$. Then, for every $\epsilon > 0$, there exists $\nu \in \mathcal{M}_0(Q)$ such that*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \epsilon \text{ and } \nu = w_t - \Delta_p w \text{ in } \mathcal{D}'(Q) \tag{5.1.2}$$

where $w \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Note that the function w is constructed as the truncation of a nonlinear potential of μ.

We will argue by density for proving the existence of a solution, so that we need the following preliminary result.

PROPOSITION 5.2. *Given $\mu \in \mathcal{M}_0(Q) \cap L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $u \in W$ be the (unique) weak solution of*

$$\begin{cases} b(u)_t - \Delta_p u = \mu & \text{in } (0, T) \times \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ b(u(0, x)) = b(u_0) & \text{in } \Omega, \end{cases} \tag{5.1.3}$$

Then

$$\mathrm{cap}_p(\{|b(u)| > k\}) \leq C \max\left\{\frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}}\right\} \quad \forall k \geq 1, \tag{5.1.4}$$

where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}_0(Q)}$, $\|b(u_0)\|_{L^1(\Omega)}$, and p.

PROOF. We still use the notations introduced in Section 1.12, in particular, we consider the condition $p > \frac{2N+1}{N+1}$, for simplicity we assume in addition that $\mu \geq 0$ and $b(u_0) \geq 0$, hence, we have $u \geq 0$ (the case $\mu \leq 0$ can be obtained similarly). Actually, the proof will be split into three parts, we begin with the first one to obtain the basic estimates.

Step 1. Estimates of $T_k(b(u))$ in the space $L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$. For every $\tau \in \mathbb{R}$, let $\overline{T}_k(r) = \int_0^r T_k(s)ds$. We recall that if $u \in W$, then u is a weak solution of (5.1.3) if

$$\int_0^t \langle b(u)_t, v\rangle dt + \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla v \; dxdt = \int_0^t \langle \mu, v\rangle dt, \quad \forall v \in W, \tag{5.1.5}$$

where $\langle \cdot, \cdot\rangle$ denotes the duality between V and V'. Note that, if $\mu \in \mathcal{M}_0(Q) \cap L^{p'}(0, T; W^{-1,p'}(\Omega))$, then (5.1.5) holds for every $v \in L^p(0, T; V)$, and we have

$$\int_s^t \langle b(u)_t, \psi'(u)\rangle dt = \int_\Omega \psi(b(u)(t))dx - \int_\Omega \psi(b(u)(s))dx, \tag{5.1.6}$$

for every $s, t \in [0, T]$ and every function $\psi : \mathbb{R} \to \mathbb{R}$ such that ψ' is Lipschitz continuous and $\psi'(0) = 0$. Now we choose as test function $T_k(b(u))$ in (5.1.5) and using (5.1.6) with $\psi = \overline{T}_k$, $s = 0$ and $t = r$ to get

$$\int_\Omega \overline{T}_k(b(u))(r)dx + \int_0^r \int_\Omega a(t, x, \nabla u) \cdot \nabla T_k(b(u))dxdt \leq k\|\mu\|_{\mathcal{M}_0(Q)} + \int_\Omega \overline{T}_k(b(u_0))dx.$$

Let $E_k = \{(t, x) : |b(u)| \leq k\}$, and observing $\frac{T_k(s)^2}{2} \leq \overline{T}_k(s) \leq k|s|$, $\forall s \in \mathbb{R}$, we have

$$\int_\Omega \frac{|T_k(b(u))(r)|^2}{2}dx + \int_0^r \int_\Omega \chi_{E_k} b'(u)a(t, x, \nabla u) \cdot \nabla u \; dxdt \leq k(\|\mu\|_{\mathcal{M}_0(Q)} + \|b(u_0)\|_{L^1(\Omega)}), \tag{5.1.7}$$

for any $r \in [0, T]$. In particular, we deduce

$$\|T_k(b(u))\|^2_{L^\infty(0,T;L^2(\Omega))} \le 2kM, \tag{5.1.8}$$

and from assumption (5.2.2), we have

$$\alpha \int_{E_k} b'(u)|\nabla u|^p dxdt \le \int_0^r \int_\Omega \chi_{E_k} b'(u) a(t,x,\nabla u)\cdot \nabla u \; dxdt \le kM.$$

Note that

$$\begin{aligned}\int_{E_k} b'(u)|\nabla u|^p dxdt &= \int_{E_k} b'(u)|b'^{-1}\nabla b(u)|^p dxdt \\ &= \int_{E_k} \frac{1}{(b')^{p-1}}|\nabla b(u)|^p dxdt \ge \int_0^r \int_\Omega \frac{1}{(b_1)^{p-1}}|\nabla T_k b(u)|^p dxdt.\end{aligned}$$

Then,

$$\|T_k(b(u))\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le CkM, \tag{5.1.9}$$

where

$$C = \frac{b_1^{p-1}}{\alpha} \quad \text{and} \quad M = \|\mu\|_{\mathcal{M}_0(Q)} + \|b(u_0)\|_{L^1(\Omega)}. \tag{5.1.10}$$

Step 2. Estimates in W. Note that in virtue of **[L, P]**, any function $z \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ is a solution of the backward problem

$$\begin{cases} -z_t - \Delta_p z = -2\Delta_p T_k(b(u)) & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z = T_k(b(u)) & \text{on } \{T\}\times\Omega. \end{cases} \tag{5.1.11}$$

We can choose z as test function in (5.1.11) and integrate t between τ and T. Since we have from Young's inequality

$$\int_\Omega \frac{[z(\tau)]^2}{2}dx + \frac{1}{2}\int_\tau^T \int_\Omega b'(u)|\nabla z|^p dxdt \le \int_\Omega \frac{[T_k(b(u))(T)]^2}{2}dx + C\int_\tau^T \int_\Omega b'(u)|\nabla u|^p dxdt$$

we deduce, using also (5.1.6) with $r = T$

$$\int_\Omega \frac{[z(\tau)]^2}{2}dx + \frac{1}{2}\int_\tau^T \int_\Omega b'(u)|\nabla z|^p dxdt \le Ck(\|\mu\|_{\mathcal{M}_0(Q)} + \|b(u_0)\|_{L^1(\Omega)}) = CkM,$$

this implies the estimate for z

$$\|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le CkM. \tag{5.1.12}$$

Since by the definition of V (i.e. $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$), we have

$$\|z\|^p_{L^p(0,T;V)} \le C(\|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|z\|^p_{L^p(0,T;L^2(\Omega))}).$$

Then we have from (5.1.12) that

$$\|z\|_{L^p(0,T;V)} \le C[(kM)^{\frac{1}{p}} + (kM)^{\frac{1}{2}}], \tag{5.1.13}$$

using the equation (5.1.11), we obtain

$$\|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C(\|z\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))} + \|T_k(b(u))\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))}),$$

hence, we get from (5.1.9) and (5.1.12)

$$\|z\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C(kM)^{\frac{1}{p'}}. \tag{5.1.14}$$

Putting together (5.1.13) and (5.1.14), we have the result

$$\|z\|_W \le C \max\left\{(kM)^{\frac{1}{p}}, (kM)^{\frac{1}{p'}}\right\}, \tag{5.1.15}$$

where M is the constant defined in (5.1.10).

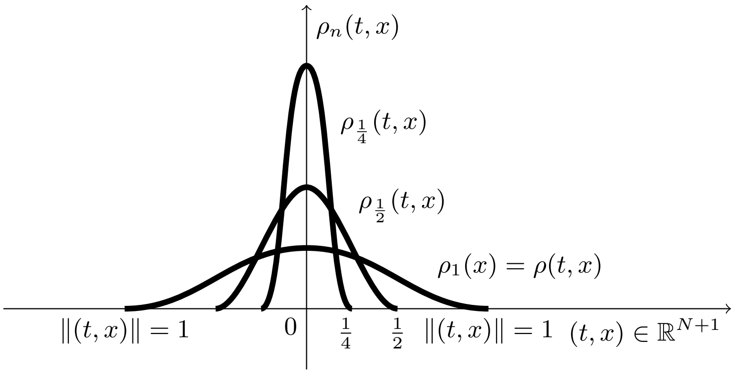


FIGURE 17. Example of mollifiers (ρ_n)

Step 3. Proof completed. Obtaining the energy inequality (5.1.15) was the main step in order to prove the estimate of the capacity (5.1.4). It should be noticed that we assume that $\mu \geq 0$ to obtain $b(u)_t - \Delta_p u \geq 0$, $u \geq 0$ in Q and the following inequality holds

$$(T_k(b(u)))_t - \Delta_p T_k(b(u)) \geq 0. \tag{5.1.16}$$

Indeed, one can choose $T'_{k,\eta}(b(u))\varphi$ (see Section 5.4) in (5.1.5) where $\varphi \in C_c^\infty(Q)$ and $\varphi \geq 0$, using this time $\mu \geq 0$, with the fact that $T_{k,\eta}(s)$ is concave for $s \geq 0$,

$$-\int_0^T \varphi_t T_{k,\eta}(b(u))dt + \int_Q b'(u)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi S_{k,\eta}(u)\ dxdt \geq 0,$$

which yields (5.1.16) as η goes to 0. Therefore, the combination of (5.1.11) and (5.1.16) gives

$$-z_t - \Delta_p z \geq -(T_k(b(u)))_t - \Delta_p T_k(b(u)). \tag{5.1.17}$$

We are left to prove that $z \geq T_k(b(u))$ a.e. in Q, in particular, $z \geq k$ a.e. on $\{b(u) > k\}$. This is done by means of $(z - T_k(b(u)))^-$ in both sides of (5.1.17), and since z and $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$. Indeed we have u has a unique cap_p quasi-continuous representative (recall that, u belongs to W); hence, the set $\{b(u) > k\}$ is cap_p quasi-open, and its capacity can be estimated with (1.12.3). So that

$$\mathrm{cap}_p(\{|b(u)| > k\}) \leq \left\|\frac{z}{k}\right\|_W .$$

Using (5.1.15) and by means that the result is also true for $\mu \leq 0$, we conclude (5.1.4). □

Now, We consider a sequence of mollifiers (ρ_n) such that for any $n \geq 1$,

$$\rho_n \in C_c^\infty(\mathbb{R}^{N+1}),\ \mathrm{Supp}\ \rho_n \subset B_{\frac{1}{n}}(0),\ \rho_n \geq 0 \text{ and } \int_{\mathbb{R}^{N+1}} \rho_n = 1. \tag{5.1.18}$$

EXAMPLE. Consider the mollifier (ρ_n) of nonnegative $C^\infty-$functions on $\mathbb{R}^{N+1}$ defined by

$$\rho_n(t,x) = \frac{1}{n}\rho(\frac{x}{n},\frac{t}{n}),\quad \mathrm{Supp}\ \rho_n = \{(t,x) \in \mathbb{R}^{N+1} : |(t,x)| \leq 1\} \text{ and } \int_{\mathbb{R}^{N+1}} \rho_n(t,x)dxdt = 1$$

where $\rho(t,x)$ is a nonnegative $C^\infty-$functions on $\mathbb{R}^{N+1}$ satisfying

$$\mathrm{Supp}\ \rho = \{(t,x) \in \mathbb{R}^{N+1} : \|(t,x)\| \leq 1\} \text{ and } \int_{\mathbb{R}^{N+1}} \rho(t,x)dxdt = 1.$$

For example, we can take

$$\rho(t,x) = \begin{cases} k\ \exp(\frac{1}{\|(t,x)\|^2-1}) & \text{for } \|(t,x)\| < 1, \\ 0 & \text{for } \|(t,x)\| \geq 1. \end{cases}$$

Given $\mu \in \mathcal{M}_0(Q)$, we define μ_n as a convolution $\rho_n * \mu$ for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ by

$$\mu_n(t,x) = \rho_n * \mu(t,x) = \int_Q \rho_n(t-s, x-y)d\mu(s,y). \tag{5.1.19}$$

DEFINITION 5.3. A sequence of measures (μ_n) in Q is equi-diffuse, if for every $\eta > 0$ there exists $\delta > 0$ such that

$$\operatorname{cap}_p(E) < \delta \Longrightarrow |\mu_n|(E) < \eta \quad \forall n \geq 1.$$

The following result is proved in [**PPP2**].

LEMMA 5.4. *Let ρ_n be a sequence of mollifiers on Q. If $\mu \in \mathcal{M}_0(Q)$, then the sequence $(\rho_n * \mu_n)$ is equi-diffuse.*

For any nonnegative real number, we denote by $T_k(r) = \min(k, \max(r, -k))$ the truncation function at level k. For every $r \in \mathbb{R}$, let $\overline{T}_k(z) = \int_0^z T_k(s)ds$. Finally by $\langle \cdot, \cdot \rangle$ we mean the duality between suitable spaces in which functions are involved. In particular we will consider both duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega) + L^1(Q)$, and we denote by $\omega(h, n, \delta, ...)$ any quantity that vanishes as the parameters go to their limit point.

5.2. Main assumptions and renormalized formulation

Let Ω be a bounded, open subset of $\mathbb{R}^N$, T a positive number and $Q = (0,T) \times \Omega$, we will actually consider a larger class of problems involving Leray-Lions type operators of the form $-\operatorname{div}(a(t,x,\nabla u))$ (the same argument as above still holds for more general nonlinear operators [**BMR**]), and the nonlinear parabolic problem

$$\begin{cases} b(u)_t - \operatorname{div}(a(t,x,\nabla u)) = \mu & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ b(u) = b(u_0) & \text{on } \{0\} \times \Omega, \end{cases} \tag{5.2.1}$$

where $a : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e., $a(\cdot,\cdot,\zeta)$ is measurable on Q for every ζ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q), such that the following assumptions holds

$$a(t,x,\zeta) \cdot \zeta \geq \alpha |\zeta|^p, \quad p > 1, \tag{5.2.2}$$

$$|a(t,x,\zeta)| \leq \beta [L(t,x) + |\zeta|^{p-1}], \tag{5.2.3}$$

$$[a(t,x,\zeta) - a(t,x,\eta)] \cdot (\zeta - \eta) > 0, \tag{5.2.4}$$

for almost every (t,x) in Q, for every ζ, η in $\mathbb{R}^N$, with $\zeta \neq \eta$, where α and β are two positive constants, and L is a nonnegative function in $L^{p'}(Q)$.
In all the following, we assume that $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function which satisfies

$$0 < b_0 \leq b'(s) \leq b_1 \quad \forall s \in \mathbb{R} \text{ and } b(0) = 0, \tag{5.2.5}$$

$$u_0 \text{ is a measurable function in } \Omega \text{ such that } b(u_0) \in L^1(\Omega), \tag{5.2.6}$$

and that μ is a diffuse measure, i.e.,

$$\mu \in \mathcal{M}_0(Q). \tag{5.2.7}$$

Let us give the notion of renormalized solution for parabolic problem (5.2.1) using a different formulation, we recall that the following definition is the natural extension of the one given in [**BPR**] for diffuse measures.

DEFINITION 5.5. Let $\mu \in \mathcal{M}_0(Q)$. A measurable function u defined on Q is a renormalized solution of problem (5.2.1) if $T_k(b(u)) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$, and if there exists a sequence (λ_k) in $\mathcal{M}_0(Q)$ such that

$$\lim_{k \to \infty} \|\lambda_k\|_{\mathcal{M}_0(Q)} = 0, \tag{5.2.8}$$

and

$$
\begin{aligned}
&-\int_Q T_k(b(u))\varphi_t dxdt + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi dxdt \\
&\qquad = \int_Q \varphi d\mu + \int_Q \varphi d\lambda_k + \int_\Omega T_k(b(u_0))\varphi(0,x)dx
\end{aligned} \tag{5.2.9}
$$

for every $k > 0$ and $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Remark 5.6. Note that

(i) Equation (5.2.9) implies that $(T_k(b(u)))_t - \mathrm{div}(a(t,x,\nabla u))$ is a bounded measure, and since $T_k(b(u)) \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\mu_0 \in \mathcal{M}_0(Q)$ this means that

$$
(T_k(b(u)))_t - \mathrm{div}(a(t,x,\frac{1}{b'(u)}\nabla T_k(b(u)))) = \mu + \lambda^k \text{ in } \mathcal{M}_0(Q). \tag{5.2.10}
$$

(ii) Thanks to a result of [**PPP2**], the renormalized solution of problem (5.2.1) turns out to coincide with the renormalized solution of the same problem in the sense of [**BPR**] (see Proof of the Theorem 5.9 bellow).

(iii) For every $\varphi \in W^{1,\infty}(Q)$ such that $\varphi = 0$ on $(\{T\}\times\Omega)\cup((0,T)\times\partial\Omega)$, we can use φ as test function in (5.2.9) or in the approximate problem.

(iv) A remark on the assumption (5.2.5) is also necessary. As one could check later, due essentially to the presence of the term g (dependent on t) in the formulation of the renormalized solution (i.e, the term with μ) in Definition 5.5 , we are forced to assume $b'(s) \geq b_0 > 0$. We conjecture that this assumption is only technical to prove the equivalence and could be removed in order to deal with more general elliptic-parabolic problems [**AHL, AW, CW**].

5.3. The formulation does not depend on the decomposition of the measure

As we said before, for every measure $\mu \in \mathcal{M}_0(Q)$, there exist a decomposition (f,g,χ) not uniquely determined such that $f \in L^1(Q)$, $g \in L^p(0,T;V)$ and $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ with

$$
\mu = f + g + \chi \text{ in } \mathcal{D}'(Q).
$$

It is not known whether if every measure which can be decomposed in this form is diffuse. However, in [**PPP2**] we have the following result

Lemma 5.7. *Assume that $\mu \in \mathcal{M}(Q)$ satisfies* (5.1.1)*, where $f \in L^1(Q)$, $g \in L^p(0,T;V)$ and $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. If $g \in L^\infty(Q)$, then μ is diffuse.*

Proof. See [**PPP2**], Proposition 3.1. □

Recall the notion of renormalized solution in the sense of [**BPR**].

Definition 5.8. Let $\mu \in \mathcal{M}_0(Q)$. A measurable function defined on Q is a renormalized solution of problem (5.2.1) if

$$
b(u) - g \in L^\infty(0,T;L^1(\Omega)), \quad T_k(b(u)-g) \in L^p(0,T;W_0^{1,p}(\Omega)), \quad \forall k > 0, \tag{5.3.1}
$$

$$
\lim_{h\to\infty} \int_{\{h\leq|b(u)-g|\leq h+1\}} |\nabla u|^p dxdt = 0, \tag{5.3.2}
$$

and for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$
\begin{aligned}
&-\int_Q S(b(u)-g)\varphi_t dxdt + \int_Q a(t,x,\nabla u)\cdot\nabla(S'(b(u)-g)\varphi)dxdt \\
&= \int_Q fS'(b(u)-g)\varphi dxdt + \int_Q G\cdot\nabla(S'(b(u)-g)\varphi)dxdt + \int_\Omega S(b(u_0))\varphi(0,x)dx,
\end{aligned} \tag{5.3.3}
$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Finally, we conclude by proving that Definition 5.5 imply that u is a renormalized solution in the sense of Definition 5.8, this proves that the formulations are actually equivalent.

THEOREM 5.9. *Let μ be splitted as in* (5.1.1), *namely*

$$\mu = f - \operatorname{div}(G) + g, \quad f \in L^1(Q),\ G \in L^{p'}(Q) \text{ and } g \in L^p(0,T;V).$$

Then, If u satisfies Definition 5.5, then u satisfies Definition 5.8.

PROOF. We split the proof in two steps

Step 1. Let $v = T_k(b(u) - g)$, we have $v \in L^p(0,T;V)$. Moreover, using the decomposition of μ in (5.1.1), and integrating by parts the term with g, we have

$$\begin{aligned}
&-\int_Q v\varphi_t \, dxdt + \int_Q \frac{1}{b'(u)} a(t,x,\nabla T_k(b(u))) \cdot \nabla\varphi \, dxdt \\
&= \int_Q f\varphi \, dxdt + \int_Q G \cdot \nabla\varphi \, dxdt + \int_Q \varphi d\lambda_k + \int_\Omega T_k(b(u_0))\varphi(0,x)dx
\end{aligned}$$

for every $\varphi \in C_c^\infty([0,T] \times \Omega)$. Observe that for every $\varphi \in W^{1,\infty}(Q)$ the above equality remains true. We can choose $\varphi(t,x)$ such that

$$\varphi(t,x) = \zeta(t,x)\frac{1}{h}\int_t^{t+h} \varphi(v(s,x))ds,$$

where $\zeta \in C_c^\infty([0,T] \times \overline{\Omega})$, $\zeta \geq 0$, $\zeta\psi(0) = 0$ on $(0,T) \times \partial\Omega$, and ψ is Lipschitz nondecreasing function. This clearly implies from **[BP1]**, Lemma 2.1 that

$$\begin{aligned}
&\liminf_{h\to 0} \left\{ -\int_Q (v - T_k(b(u_0))) \left(\zeta\frac{1}{h}\int_t^{t+h} \psi(v)ds \right)_t dxdt \right\} \\
&\geq -\int_Q \left(\int_0^t \psi(r)dr \right) \zeta_t dxdt - \int_\Omega \left(\int_0^{T_k(b(u_0))} \psi(r)dr \right) \zeta(0,x)dx.
\end{aligned}$$

Indeed, since ψ is bounded, we have

$$|\int_Q \psi d\lambda_k| \leq \|\zeta\|_\infty \|\psi\|_\infty \|\lambda_k\|_{\mathcal{M}_0(Q)},$$

and since ψ is Lipschitz, we have $\psi(v) \in L^p(0,T;W_0^{1,p}(\Omega))$. Notice that $(\psi(v))_h$ converges to $\psi(v)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ and weakly–$*$ in $L^\infty(Q)$. So that, as $h \to 0$,

$$\begin{aligned}
&-\int_Q \left(\int_0^r \psi(r)dr \right) \zeta_r dxdt + \int_Q a(t,x,\nabla T_k(u)) \cdot \nabla(\psi(r)\zeta)dxdt \\
&\leq \int_Q f\psi(v)\zeta dxdt + \int_Q G \cdot \nabla(\psi(v)\zeta)dxdt \\
&+ \int_\Omega \left(\int_0^{T_k(b(u_0))} \psi(r)dr \right) \zeta(0,x)dx + \|\zeta\|_\infty \|\psi\|_\infty \|\lambda_k\|_{\mathcal{M}_0(Q)},
\end{aligned} \tag{5.3.4}$$

for every ψ Lipschitz and nondecreasing. In order to obtain the reverse inequality, we only need to take

$$\varphi(t,x) = \{(t,x)\frac{1}{h}\int_{t-h}^t \psi(\tilde{v}(s,x))ds\}$$

where $\tilde{v}(t,x) = v(t,x)$ when $t \geq 0$ and $\tilde{v} = U_j$ when $t < 0$, being $U_j \in C_c^\infty(\Omega)$ such that $U_j \to T_k(b(u_0))$ strongly in $L^1(\Omega)$. Thus, using **[BP1]**, Lemma 2.3, we obtain

$$\begin{aligned}
&\liminf_{h\to 0} \left\{ -\int_Q (v - T_k(b(u_0)))(\zeta\frac{1}{h}\int_{t-h}^t \psi(v)ds)_t dxdt \right\} \\
&\leq -\int_Q \left(\int_0^r \psi(r)dr \right) \zeta_t dxdt - \int_\Omega \left(\int_0^{U_j} \psi(r)dr \right) \zeta(0,x)dx \\
&- \int_\Omega (T_k(b(u_0)) - U_j)\zeta(0,x)dx
\end{aligned}$$

Recalling that $\tilde{v} \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, when $h \to 0$, we can pass to the limit in the other terms as before, and we observe that

$$
\begin{aligned}
&-\int_Q \left(\int_0^v \psi(r)dr\right)\zeta_t dxdt + \int_Q a(t,x,\nabla u)\cdot\nabla(\psi(v)\zeta)dxdt \\
&\geq \int_Q f\psi(v)\zeta dxdt + \int_Q G\cdot\nabla(\psi(v)\zeta)dxdt + \int_\Omega \left(\int_0^{U_0}\psi(r)dr\right)\zeta(0,x)dx \\
&+ \int_\Omega (T_k(b(u_0)) - U_j)\psi(U_j)\zeta(0,x)dx - \|\zeta\|_\infty\|\psi\|_\infty\|\lambda_k\|_{\mathcal{M}_0(Q)}
\end{aligned}
$$

Hence, from $U_j \to T_k(b(u_0))$, we have

$$
\begin{aligned}
&-\int_Q \left(\int_0^v \psi(r)dr\right)\zeta_t dxdt + \frac{1}{b'(u)}\int_Q a(t,x,\nabla T_k(b(u)))\cdot\nabla(\psi(r)\zeta)dxdt \\
&\geq \int_Q f\psi(v)\zeta dxdt + \int_Q G\cdot\nabla(\psi(v))dxdt + \int_\Omega \left(\int_0^{T_k(b(u_0))}\psi(r)dr\right)\zeta(0,x)dx \\
&- \|\zeta\|_\infty\|\psi\|_\infty\|\lambda_k\|_{\mathcal{M}_0(Q)}.
\end{aligned} \tag{5.3.5}
$$

Using equality (5.3.4) with ($S \in W^{2,\infty}(\mathbb{R})$ and $\psi = \int_0^s (S''(t))^+dt$) and equality (5.3.5) with ($\psi = \int_0^s (S''(t))^-dt$), we easily deduce by subtracting the two inequalities (observe that $S'(s) = \int_0^s (S''(t)^+ - S''(t)^-)dt$) that

$$
\begin{aligned}
&-\int_Q S(v)\zeta_t dxdt + \int_Q a(t,x,\nabla u)\cdot\nabla(S'(v)\zeta)dxdt \\
&\leq \int_Q fS'(v)\zeta dxdt + \int_Q G\cdot\nabla(S'(v)\zeta)dxdt \\
&+ \int_\Omega S(T_k(b(u_0)))\zeta(0,x)dx + 2\|\zeta\|_\infty\|S'\|_\infty\|\lambda_k\|_{\mathcal{M}_0(Q)},
\end{aligned} \tag{5.3.6}
$$

for every $S \in W^{2,\infty}(\mathbb{R})$ and for every nonnegative ζ.

Step 2. Let us use $S'(\Theta_h(s))$ in (5.3.6) such that $\Theta_h = T_1(s - T_h(s))$ and $\zeta = \zeta(t)$. Then we easily obtain by setting $R_h(s) = \int_0^s \Theta_h(\zeta)d\zeta$,

$$
\begin{aligned}
&-\int_Q R_h(T_k(b(u)) - g)\zeta_t dxdt + \int_{\{h<|b(u)-g|<h+k\}} a(t,x,\nabla u)\cdot\nabla(T_k(b(u)) - g)\zeta dxdt \\
&\leq \int_Q f\Theta_h(T_k(b(u)) - g)\zeta dxdt + \int_{\{h<|b(u)-g|<h+k\}} G\cdot\nabla(T_k(b(u) - g))dxdt \\
&+ \int_\Omega R_h(T_k(b(u_0)))\zeta(0,x)dx + 2\|\zeta\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}.
\end{aligned}
$$

Moreover, we can use young's inequality, assumption (5.2.2) and (5.2.3) to get

$$
\begin{aligned}
&-\int_Q R_h(T_k(b(u) - g))\zeta_t dxdt + \int_{\{h<|b(u)-g|<h+1\}} b'(u)|\nabla T_k(b(u))|^p\zeta dxdt \\
&\leq \int_Q f\Theta_h(T_k(b(u)) - g)\zeta dxdt + C\int_{\{h<|b(u)-g|<h+1\}} (|G|^{p'} + |g|^p + |L|^{p'})\zeta dxdt \\
&+ \int_\Omega R_h(T_k(b(u_0)))\zeta(0,x)dx + 2\|\zeta\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)},
\end{aligned}
$$

Now, letting $k \to \infty$, thanks to (5.2.8) and Fatou's Lemma, we deduce

$$
\begin{aligned}
&-\int_Q R(b(u) - g)\zeta_t dxdt + \alpha\int_{\{h<|b(u)-g|<h+1\}} b'(u)|\nabla u|^p dxdt \\
&\leq \int_Q f\Theta_h(u - g)\zeta dxdt + C\int_{\{h<|b(u)-g|\leq h+1\}} (|G|^{p'} + |g|^p + |L|^{p'})\zeta dxdt \\
&+ \int_\Omega R_h(b(u_0))\zeta(0,x)dx
\end{aligned}
$$

Consider $\zeta = 1 - \frac{1}{\epsilon}T_\epsilon(t-\tau)^+$, for $\tau \in (0,T)$, and letting $\epsilon \to 0$, we claim that the estimate of $b(u) - g$ in $L^\infty(0,T;L^1(\Omega))$ is valid. By repeating the argument for the nonincreasing $\zeta_\epsilon \in C_c^\infty([0,T])$, we are allowed to pass to the limit $\zeta_\epsilon \to 1$ to prove that

$$\begin{aligned} &b_0\alpha \int_{\{h<|b(u)-g|<h+1\}} |\nabla u|^p dxdt \\ &\le \int_{\{|b(u)-g|>h\}} |f| dxdt + C\int_{\{h<|b(u)-g|<h+1\}} (|G|^{p'} + |g|^p + |L|^{p'})\zeta dxdt + \int_{\{|b(u_0)|>h\}} b(u_0)dx, \end{aligned}$$

which implies (5.3.2). Finally, by using $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support, $\zeta \in C_c^\infty([0,T]\times\Omega)$ and the regularity (5.3.1), we can easily deduce (5.3.3) by passing to the limit in (5.3.6) and using (5.2.8). $\square$

5.4. Existence of renormalized solutions

Now we are ready to prove our main result. Some of the reasoning is based on the ideas developed in [**BPR**] (see also [**DPP, PPP2, Po1**]). First we have to prove the existence of renormalized solution for problem (5.2.1).

Theorem 5.10. *Under assumptions* (5.2.1) – (5.2.7)*, there exists at least a renormalized solution u of problem* (5.2.1)*.*

Proof. We first introduce the approximate problem. For $n \ge 1$ fixed, we define

$$b_n(s) = b(T_{\frac{1}{n}}(s)) + ns \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R}, \tag{5.4.1}$$

$$u_0^n \in C_0^\infty(\Omega): \quad b_n(u_0^n) \to b(u_0) \text{ in } L^1(\Omega) \text{ as n tends to } +\infty. \tag{5.4.2}$$

We consider a sequence of mollifiers (ρ_n), and we define the convolution $\rho_n * \mu$ for every $(t,x) \in Q$ by

$$\mu^n(t,x) = \rho_n * \mu(t,x) = \int_Q \rho_n(t-s, x-y)d\mu(s,y). \tag{5.4.3}$$

Then we consider the approximate problem of (5.2.1)

$$\begin{cases} (b_n(u_n))_t - \operatorname{div}(a(t,x,\nabla u_n)) = \mu_n & \text{in } (0,T)\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ b_n(u_n) = b_n(u_0^n) & \text{on } \{0\}\times\Omega. \end{cases} \tag{5.4.4}$$

By classical results [**L**], we can find a nonnegative weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ for problem (5.4.4). Our aim is to prove that a subsequence of these approximate solutions (u_n) converges increasingly to a measurable function u, which is a renormalized solution of problem (5.2.1). We will divide the proof into several steps. We present a self-contained proof for the sake of clarity and readability.

Step 1. Basic estimates. Choosing $T_k(b_n(u_n) - g_n)$ as a test function in (5.4.4), we have

$$\begin{aligned} &\int_\Omega \overline{T}_k(b_n(u_n)-g_n)dx + \int_0^t\int_\Omega a(x,s,\nabla u_n)\cdot\nabla T_k(b_n(u_n)-g_n)dxds \\ &= \int_0^t\int_\Omega f_nT_k(b_n(u_n)-g_n)dxdt + \int_0^t\int_\Omega G_n\cdot\nabla T_k(b_n(u_n)-g_n)dxds + \int_\Omega \overline{T}_k(b_n(u_0^n))dx, \end{aligned} \tag{5.4.5}$$

for almost every t in $(0,T)$, and where $\overline{T}_k(r) = \int_0^r T_k(s)ds$. It follows from the definition of $\overline{T}_k$, assumptions (5.2.2) – (5.2.3) and (5.2.6) that

$$\begin{aligned} &\int_\Omega \overline{T}_k(b_n(u_n)-g_n)dx + \alpha\int_{\{|b_n(u_n)-g_n|\le k\}} b_n'(u_n)|\nabla u_n|^p dxds \\ &\le k\|\mu_n\|_{L^1(Q)} + \beta\int_{\{|b_n(u_n)-g_n|\le k\}} L(x,s)|\nabla g_n|dxds \\ &+ \beta\int_{\{|b_n(u_n)-g_n|\le k\}} |\nabla u_n|^{p-1}|\nabla g_n|dxds + k\|b_n(u_0^n)\|_{L^1(\Omega)} \end{aligned} \tag{5.4.6}$$

Then, from (5.2.5) and young's inequality

$$\begin{aligned}&\int_\Omega \overline{T}_k(b_n(u_n)-g_n)dx+\frac{\alpha}{2}\int_{\{|b_n(u_n)-g_n|\le k\}} b_n'(u_n)|\nabla u_n|^p dxdt\\ &\le k\|\mu_n\|_{L^1(Q)}+\beta\|L\|_{L^{p'}(Q)}\|\nabla g_n\|_{L^p(Q)}+C\|\nabla g_n\|^{p'}_{L^{p'}(Q)}+k\|b_n(u_0^n)\|_{L^1(\Omega)}\end{aligned}\tag{5.4.7}$$

where C is a positive constant. We will use the properties of $\overline{T}_k$ ($\overline{T}_k\ge 0$, $\overline{T}_k(s)\ge |s|-1$, $\forall s\in\mathbb{R}$), b_n, f_n, G_n, g_n, the boundedness of μ_n in $L^1(Q)$ and $b_n(u_0^n)$ in $L^1(\Omega)$ to have

$$b_n(u_n)-g_n \text{ is bounded in } L^\infty(0,T;L^1(\Omega))\tag{5.4.8}$$

Using Hölder inequality and (5.2.5), we deduce that (5.4.7) implies

$$T_k(b_n(u_n)-g_n) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)),\tag{5.4.9}$$

Independently of n for any $k\ge 0$.

Let us observe from **[BM, BMR]** that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support (i.e., $\mathrm{Supp}(S')\subset[-k,k]$)

$$S(b_n(u_n)-g_n) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)),\tag{5.4.10}$$

and

$$(S(b_n(u_n)-g_n))_t \text{ is bounded in } L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega)).\tag{5.4.11}$$

independently of n. In fact, thanks to (5.4.9) and Stampacchia's theorem, we easily deduce (5.4.10). To show that (5.4.11) hold true, we multiply (5.4.4) by $S'(b_n(u_n)-g_n)$ to obtain

$$\begin{aligned}(S(b_n(u_n)-g_n))_t&=\mathrm{div}(S(b_n(u_n)-g_n)a(t,x,\nabla u_n))\\ &\quad-a(t,x,\nabla u_n)\cdot\nabla S'(b_n(u_n)-g_n)+f_nS'(b_n(u_n)-g_n)\\ &\quad-\mathrm{div}(G_nS'(b_n(u_n)-g_n))+G_n\cdot\nabla S(b_n(u_n)-g_n) \text{ in } \mathcal{D}'(Q),\end{aligned}\tag{5.4.12}$$

as a consequence each term in the right hand side of (5.4.12) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ or in $L^1(Q)$, we obtain (5.4.11).

Moreover, arguing again as in **[BPR]** (see also **[BM, BMR, BR]**), there exists a measurable function u such that $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$, u belongs to $L^\infty(0,T;L^1(\Omega))$, and up to a subsequence, for any $k>0$ we have

$$\begin{cases}u_n\to u \text{ a.e. in } Q,\\ T_k(u_n)\rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\\ b_n(u_n)-g_n\to b(u)-g \text{ a.e. in } Q,\\ T_k(b_n(u_n)-g_n)\rightharpoonup T_k(b(u)-g) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\end{cases}\tag{5.4.13}$$

as n tends to $+\infty$.

Step 2. Estimates in $L^1(Q)$ on the energy term. Let ρ_n a sequence of mollifiers as in (5.1.18) and μ a nonnegative measure such that $\mu_n(t,x)=\rho_n*\mu(t,x)$. Observe that, based on Lemma 5.4 that μ_n is an equi-diffuse sequence of measures. Moreover, there exists a sequence $\mu_n\in C^\infty(Q)$ such that

$$\|\mu\|_{L^1(Q)}\le\|\mu\|_{\mathcal{M}_0(Q)},\quad \mu_n\to\mu \text{ tightly in } \mathcal{M}_0(Q).$$

Let us consider the auxiliary functions $S_{k,\eta}(s):\mathbb{R}\to\mathbb{R}$ and $h_{k,\eta}(s):\mathbb{R}\to\mathbb{R}$ that we will often use in the next chapters; this functions can be introduced in terms of $T_k(s)$ and $S_k(s)$ and defined as follows,

$$S_{k,\eta}(s)=\begin{cases}0 & \text{if } |s|\ge k+\eta\\ 1 & \text{if } |s|\le k\\ \text{affine} & \text{otherwise}\end{cases}\qquad h_{k,\eta}(s)=1-S_{k,\eta}(s)=\begin{cases}1 & \text{if } |s|\ge k+\eta\\ 0 & \text{if } |s|\le k\\ \text{affine} & \text{otherwise}\end{cases}\tag{5.4.14}$$

Let us denote by $T_{k,\eta}:\mathbb{R}\to\mathbb{R}$ the primitive function of $S_{k,\eta}$, that is

$$T_{k,\eta}(s)=\int_0^s S_{k,\eta}(\sigma)d\sigma$$

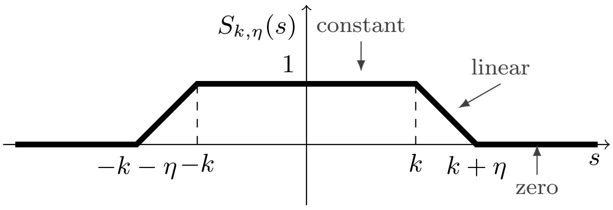


FIGURE 18. The function $S_{k,\eta}(s)$

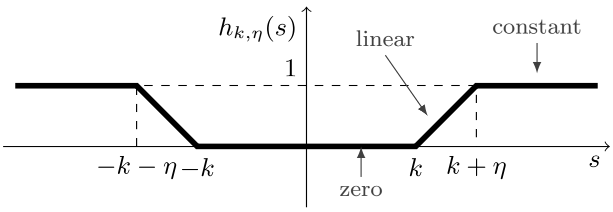


FIGURE 19. The function $h_{k,\eta}(s)$

Notice that $T_{k,\eta}(s)$ converges pointwise to $T_k(s)$ as η goes to zero and using the admissible test function $h_{k,\eta}(b(u_n))$ in (5.4.4) leads to

$$\begin{aligned}&\int_\Omega \overline{h}_{k,\eta}(b(u_n)(T))dx + \frac{1}{\eta}\int_{\{k<u_n<k+\eta\}} a(t,x,\nabla u_n)\cdot\nabla h_{k,\eta}(b(u_n))\ dxdt\\ &= \int_Q h_{k,\eta}(b(u_n))\mu_n\ dxdt + \int_\Omega \overline{h}_{k,\eta} b(u_0^n)dx,\end{aligned} \tag{5.4.15}$$

where $\overline{h}_{k,\eta}(r) = \int_0^r h_{k,\eta}(s)ds \geq 0$. Hence, using (5.4.2), (5.4.3) and dropping a nonnegative term,

$$\begin{aligned}&\frac{1}{\eta}\int_{\{k<|b(u_n)|<k+\eta\}} b'(u_n)a(t,x,\nabla u_n)\cdot\nabla u_n\ dxds\\ &\leq \int_{\{|b(u_n)|>k\}} |\mu_n|dxdt + \int_{\{|b(u_0^n)|>k\}} |b(u_0^n)|dx \leq C.\end{aligned} \tag{5.4.16}$$

Thus, there exists a bounded Radon measures λ_k^n such that, as η tends to zero

$$\lambda_k^{n,\eta} = \frac{1}{\eta}a(t,x,\nabla u_n)\cdot\nabla u_n \chi_{\{k\leq|b(u_n)|\leq k+\eta\}} \rightharpoonup \lambda_k^n \text{ weakly--* in } \mathcal{M}_0(Q). \tag{5.4.17}$$

Step 3. Equation for the truncations. We are able to prove that (5.2.9) holds true. To see that, we multiply (5.4.4) by $S_{k,\eta}(b(u_n))\xi$ where $\xi \in C_c^\infty([0,T]\times\Omega)$ to obtain

$$\begin{aligned}&T_{k,\eta}(b(u_n))_t - \operatorname{div}(S_{k,\eta}(b(u_n))a(t,x,\frac{1}{b'(u_n)}\nabla T_{k,\eta}(b(u_n))))\\ &= \mu_n + (S_{k,\eta}(b(u_n))-1)\mu_n + \frac{1}{n}a(t,x,\nabla u_n)\cdot\nabla u_n\chi_{\{k<|b(u_n)|<k+\eta\}} \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{5.4.18}$$

Passing to the limit in (5.4.18) as η tends to zero, and using the fact that $|S_{k,\eta}| \leq 1$ and (5.4.17), we deduce

$$T_k(b(u_n))_t - \operatorname{div}(a(t,x,\frac{1}{b'(u)}\nabla T_k(b(u_n)))) = \mu_n - \mu_n\chi_{\{|b(u_n)|\leq k\}} + \lambda_k^n \text{ in } \mathcal{D}'(Q). \tag{5.4.19}$$

Now, using properties of the convolution $\rho_n * \mu$ and in view of (5.4.16) – (5.4.17), we deduce that $\Lambda_k^n = -\mu_n\chi_{\{|b(u_n)|<k\}}+\lambda_k^n$ is bounded in $L^1(Q)$. Then there exists a bounded measures Λ_k such that $(-\mu_n\chi_{\{|b(u_n)|<k\}}+\lambda_k^n)_n$ converges to Λ_k weakly–* in $\mathcal{M}_0(Q)$. Therefore, using results (5.4.13) of *Step.1* and (5.4.19) we deduce that u satisfies

$$T_k(b(u))_t - \operatorname{div}(a(t,x,\nabla u)) = \mu + \Lambda_k \text{ in } \mathcal{D}'(Q). \tag{5.4.20}$$

Step 4. u is a renormalized solution. In this step, Λ_k is shown to satisfy (5.2.8). From (5.4.16) and (5.4.17) we deduce

$$\begin{aligned}\|\Lambda_k^n\|_{L^1(Q)} &= \| -\mu_n\chi_{\{|b(u_n)|>k\}} + \lambda_k^n\|_{L^1(Q)}\\ &\le 2\int_{\{|b(u_n)|>k\}} |\mu_n|dxdt + \int_{\{|b(u_0^n)|>k\}} |b(u_0^n)|dx.\end{aligned} \tag{5.4.21}$$

Since

$$\|\lambda_k\|_{\mathcal{M}_0(Q)} \le \liminf_{n\to+\infty} \|\mu_n\chi_{\{|b(u_n)|>k\}} + \lambda_k^n\|_{\mathcal{M}_0(Q)},$$

the sequence (μ_n) is equi-diffuse, and the function $b(u_0^n)$ converges to $b(u_0)$ strongly in $L^1(\Omega)$, we deduce from Proposition 5.2 and (5.4.21) that $\|\Lambda_k\|_{\mathcal{M}(Q)}$ tends to zero as k tends to infinity, then we obtain (5.2.8), and hence, u is a renormalized solution. □

5.5. Uniqueness of renormalized solutions

This section is devoted to establish the uniqueness of the renormalized solution. As we already said, due to the presence of both the general monotone operator associated to a and the nonlinearity of the term b, a standard approach (see for instance [**DPP**]) does not apply here. To overcome this difficulty, we are going to exploit the idea of [**PPP2**] for which the uniqueness result comes from the following comparison principle.

THEOREM 5.11. *Let u_1, u_2 be two renormalized solutions of problem* (5.2.1) *with data* $(b(u_0^1), \mu_1)$ *and* $(b(u_0^2), \mu_2)$ *respectively. Then, we have*

$$\int_\Omega (b(u_1) - b(u_2))^+(t)dx \le \|b(u_0^1) - b(u_0^2)\|_{L^1(\Omega)} + \|(\mu_1 - \mu_2)^+\|_{\mathcal{M}(Q)} \tag{5.5.1}$$

for almost every $t \in [0,T]$. In particular, if $b(u_0^1) \le b(u_0^2)$ and $\mu_1 \le \mu_2$ (in the case of measures), we have $u_1 \le u_2$ a.e. in Q. As a consequence, there exists at least one renormalized solution of problem (5.2.1).

PROOF. Let $\lambda_{k_1}, \lambda_{k_2}$ be the measures given by Definition 5.5 corresponding to $b(u_1), b(u_2)$, we can extend the class of test functions

$$\begin{aligned}&-\int_Q (T_k(b(u_1)) - T_k(b(u_2))v_t dxdt + \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla v dxdt\\ &= \int_Q vd(\mu_1-\mu_2) + \int_Q vd\lambda_{k,1} - \int_Q vd\lambda_{k,2} + \int_\Omega (T_k(b(u_0^1)) - T_k(b(u_0^2)))v(0,x)dx,\end{aligned}$$

for every $v \in W \cap L^\infty(Q)$, such that $v(T) = 0$. Consider the function

$$\omega_h(t,x) = \frac{1}{h}\int_t^{t+h} \frac{1}{\epsilon} T_\epsilon(T_k(b(u_1)) - T_k(b(u_2)))^+(s,x)ds.$$

Given $\zeta \in C_c^\infty([0,T))$, $\zeta \ge 0$, take $v = \omega_h\zeta$ as test function. Observe that both ω_h and $(\omega_h)_t$ belong to $L^p(0,T;V) \cap L^\infty(Q)$ for $h > 0$ sufficiently small, hence $\omega_h \in W \cap L^\infty(Q)$. Moreover, we have

$$\omega_h \to \frac{1}{\epsilon}T_\epsilon(T_k(b(u_1)) - T_k(b(u_2)))^+ \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

Using that $0 \le \omega_h \le 1$ almost everywhere, hence $0 \le \omega_h \le 1$ cap_p quasi-everywhere [**DPP**], we have

$$\begin{aligned}&-\int_Q [(T_k(b(u_1)) - T_k(b(u_2)) - (T_k(b(u_0^1)) - T_k(b(u_0^2)))](\omega_h\zeta)_t dxdt\\ &+ \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla\omega_h\zeta dxdt\\ &\le \|\zeta\|_\infty(\|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)}).\end{aligned} \tag{5.5.2}$$

Using the monotonicity of $T_\epsilon(s)$ and [**BP1**], Lemma 2.1, we have

$$\begin{aligned}&\liminf_{h\to 0}\{-\int_Q [(T_k(b(u_1)) - T_k(b(u_2)) - (T_k(b(u_0^1)) - T_k(b(u_0^2)))](\omega_h\zeta_t)dxdt\}\\ &\ge -\int_Q \tilde\Theta_\epsilon(T_k(b(u_1))\zeta_t dxdt - \int_\Omega \tilde\Theta_\epsilon(T_k(b(u_0^1)) - T_k(b(u_0^2))\zeta(0)dx\end{aligned}$$

where $\tilde{\Theta}_\epsilon(s) = \int_0^s \frac{1}{\epsilon} T_\epsilon(r)^+ dr$. Therefore, letting $h \to 0$ in (5.5.2), we obtain

$$
\begin{aligned}
&- \int_Q \tilde{\Theta}_\epsilon(T_k(b(u_1)) - T_k(b(u_2))\zeta_t dxdt \\
&+ \frac{1}{\epsilon} \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2)) \cdot \nabla T_\epsilon(T_k(b(u_1)) - T_k(b(u_2))\zeta dxdt \\
&\leq \int_\Omega \tilde{\Theta}_\epsilon(T_k(b(u_0^1)) - T_k(b(u_0^2))\zeta(0)dx + \|\zeta\|_\infty(\|(\mu_1 - \mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)}).
\end{aligned}
$$

Using (5.2.4) and letting $\epsilon \to 0$, we deduce

$$
\begin{aligned}
- \int_Q (T_k(b(u_1)) - T_k(b(u_2))^+ \zeta_t dxdt \leq& \int_\Omega (T_k(b(u_0^1)) - T_k(b(u_0^2))^+ \zeta(0)dx \\
&+ \|\zeta\|_\infty(\|(\mu_1 - \mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)})
\end{aligned}
$$

and letting $k \to \infty$, we obtain, thanks to (5.2.8),

$$
- \int_Q (b(u_1) - b(u_2))^+ \zeta_t dxdt \leq \|\zeta\|_\infty(\|(b(u_0^1) - b(u_0^2)^+\|_{L^1(\Omega)} + \|(\mu_1 - \mu_2)^+\|_{\mathcal{M}(Q)})
$$

for every nonnegative $\zeta \in C_c^\infty([0,T))$. Of course, the same inequality holds for any $\zeta \in W^{1,\infty}(0,T)$ with compact support in $[0,T)$. Take then $\zeta(t) = 1 - \frac{1}{\epsilon} T_\epsilon(t-\tau)^+$, where $\tau \in (0,T)$; since $b(u_1),\ b(u_2) \in L^\infty(0,T;L^1(\Omega))$, by letting $\epsilon \to 0$, we have

$$
- \int_Q (b(u_1) - b(u_2))^+ \zeta_t dxdt = \frac{1}{\epsilon} \int_\tau^{\tau+\epsilon} \int_\Omega (b(u_1) - b(u_2))^+ dxdt \to \int_\Omega (b(u_1) - b(u_2))^+(\tau) dx
$$

for almost every $\tau \in (0,T)$. Using in the right-hand side that $\|\zeta\|_\infty \leq 1$, we get (5.5.1). □

CHAPTER 6

Generalized porous medium problems with Leray–Lions operators and general measure data

Generalized porous medium equations have attracted increasing attention over the last twenty years for their applications in continuum mechanics, population dynamics and image processing [**Va**]. Under the assumption that b is a bounded, increasing C^1-function and depends only on u, the reader is referred to the Chapter 5, to the work [**BR**] for problems with data in $L^1(Q)$ and [**BPR**] for diffuse measure. It is particularly important to study the solutions u when such functions b are unbounded and depends on x and u

$$
\begin{cases} b(x,u)_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega. \end{cases} \tag{6.0.1}
$$

The case where the right-hand side belongs to $L^1(Q)$ has been studied in [**R1**], in particular for a class of nonlinear parabolic operators with continuous function Φ, the existence of renormalized solutions for problems with bounded Radon measure μ which does not charge sets of null capacity, $\mu \in \mathcal{M}_0(Q)$, using a compactness argument in the sense of [**BPR, DPP**] was proved in [**MR**]. Finally, in [**MBR**] the authors discussed problems (6.0.1) with absorption term and equi-diffuse measure and developed an existence result of renormalized solutions using the theory of [**PPP1, PPP2**] by a different type of approximations. As far as the unbounded term $b(x,u)$ is concerned, the case of general measure has not been investigated [**Pe1, Pe3**]. In this Chapter, we study the existence of the special type of distributional solutions, the so-called "renormalized solutions" for problems (6.0.1). Our results cover the case of general measures and are also new in such cases of problems. We construct an approximate sequence of solutions and we establish some a priori estimates. Then we draw a subsequence to obtain a limit function, and prove that this function is a renormalized solution. Based on the "cut-off" test functions and the "near-far from" approach we obtain a new properties, which leads to treat the singular term of the measure. We would like to mention that the proof do not based on the strong convergence of truncates and can be extended to a larger class of non-monotone operators a with respect to u. This Chapter is organized as follows. In Section 6.1, some preliminary results on capacity and basic properties on measures, the main assumptions and the definition of renormalized solution will be given. In Section 6.2, we set the a priori estimates and the existence result, while Section 6.3 is devoted to the proof of the main result. Finally, in Section 6.4 we discuss some asymptotic properties of the singular part of the measure and the proof of a capacitary estimate of the solution.

6.1. Main assumptions and renormalized solutions

We will denote, respectively, by $b_s(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\nabla_x b(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}^N$ the derivative parts of $b(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ with respect to s and to x defined, respectively, as $b_s(x,s) = \frac{\partial b}{\partial s}(x,s)$ and $\nabla_x b_x(x,s) = \frac{\partial b}{\partial x}(x,s)$ (with a slight abuse of notation, we will write $b_s, \nabla_x b$ and B every time this terms appears, instead of using its real values representations). Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, T is a positive number. We focus our attention on the well-posedness of renormalized solution for the problem

$$
\begin{cases} b(x,u)_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \end{cases} \tag{6.1.1}
$$

where $1 < p < N$, $b(x,u)$ is a unbounded function of u and $-\mathrm{div}(a(t,x,u,\nabla u))$ is the Leray-Lions operator which satisfy a polynomial growth condition with respect to u and ∇u. Moreover, assume that the following assumptions hold true

$$b(x,s),\ b_s(x,s) : \Omega \times \mathbb{R} \to \mathbb{R} \text{ and } b_x(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}^N \text{ are Carathéodory functions,} \tag{6.1.2}$$

such that for every $x \in \Omega$, $b(x,\cdot)$ is a strictly increasing C^1-function with $b(x,0) = 0$ and there exists $\lambda,\ \Lambda > 0$ and a function $B \in L^p(\Omega)$ such that

$$\lambda \le b_s(x,s) \le \Lambda \text{ for a.e. } (x,s) \in \Omega \times \mathbb{R}, \tag{6.1.3}$$

$$|\nabla_x b(x,s)| \le B(x) \text{ a.e. } x \in \Omega. \tag{6.1.4}$$

Now, let $a(t,x,s,\zeta) : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e., $a(\cdot,\cdot,s,\zeta)$ is measurable on Q for every (s,ζ) in $\mathbb{R} \times \mathbb{R}^N$, and $a(t,x,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost every (t,x) in Q) such that

$$a(t,x,s,\zeta) \cdot \zeta \ge \alpha |\zeta|^p, \quad p > 1, \tag{6.1.5}$$

for a.e. $(t,x) \in Q$, for all (s,ζ) in $\mathbb{R} \times \mathbb{R}^N$, with α is a positive constant,

$$|a(t,x,s,\zeta)| \le \beta(L(t,x) + |s|^{p-1} + |\zeta|^{p-1}), \tag{6.1.6}$$

for a.e. $(t,x) \in Q$, for any $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$, with β is a positive constant and L is a non-negative function in $L^{p'}(Q)$,

$$[a(t,x,s,\zeta) - a(t,x,s,\eta)] \cdot (\zeta - \eta) > 0, \tag{6.1.7}$$

for a.e. $(t,x) \in Q$ and for every $(s,\zeta,\eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, with $\zeta \ne \eta$.

Under these assumptions, the operator $A(u) = -\mathrm{div}(a(t,x,u,\nabla u))$ turns out to be a continuous, coercive, pseudo-monotone from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Moreover, assume that

$$u_0 \text{ is a measurable function on } \Omega \text{ such that } b(x,u_0) \in L^1(\Omega), \tag{6.1.8}$$

$$\mu \in \mathcal{M}_b(Q). \tag{6.1.9}$$

To simplify the notations, Let us define for every $p > 1$, The capacity Sobolev space

$$W = \{u \in L^p(0,T;V);\ u_t \in L^{p'}(0,T;V')\}$$

which is a Banach space endowed with the norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$, where $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$. The space S^p as

$$S^p = \{u \in L^p(0,T;W_0^{1,p}(\Omega));\ u_t \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))\}$$

endowed with its natural norm $\|u\|_{S^p} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$, it is clear that $S^p \underset{\text{inj cont}}{\hookrightarrow} C(0,T;L^1(\Omega))$ and its subspace W_2 as

$$W_2 = \{u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q);\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\}$$

endowed with its natural norm $\|u\|_{W_2} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u\|_{L^\infty(Q)} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$. Let us also define the measurable function $v = b(x,u) - g$, where u is the solution, g is the time-derivative part of μ_0, and $\tilde{\mu}_0 = \mu - g - \mu_s = f - \mathrm{div}(G)$. Moreover, we have $\nabla u \chi_{\{|v| \le k\}} = b_s^{-1}(x,u)(\nabla T_k(v) + (\nabla g - \nabla_x b(x,u))\chi_{|v| \le k})$. Let us recall some ideas contained in **[PPP2]** and essential to prove the existence of renormalized solutions. The next result shows that every function in W_2 satisfy a capacitary estimate for the $p-$capacity.

Lemma 6.1. *Let $z \in W_2$, then z admits a unique cap_p quasi-continuous representative. Moreover, we have*

$$\mathrm{cap}_p(\{|z| > k\}) \le \frac{C}{k} \max([z]_*^{\frac{1}{p}}, [z]_*^{\frac{1}{p'}}) \tag{6.1.10}$$

where $[z]_* = \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega)))} + \|z_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|z\|_{L^\infty(Q)}\|z_t^2\|_{L^1(Q)} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))}$, such that $z_t^1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $z_t^2 \in L^1(Q)$ is any decomposition of z_t, that is $z_t = z_t^1 + z_t^2$.

Proof. See [**Pe1**], Theorem 3 and Lemma 2. □

Remark 6.2. Notice that Lemma 6.1 is useful to obtain a unique cap_p quasi-continuous representative of $u \in W_2$ defined cap_p quasi-everywhere. Then

$$\mathrm{cap}_p(\{|b(x,u)| > k\}) \leq \frac{C}{k} \max(\|u\|_{W_2}^p, \|u\|_{W_2}^{p'}). \tag{6.1.11}$$

The previous lemma has also the following consequence.

Lemma 6.3. *Let $\mu \in \mathcal{M}_b(Q) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $b(x,u_0) \in L^2(\Omega)$. Then, under the assumptions* (6.1.2) – (6.1.4) *the weak solution u of* (6.1.1) *belongs to W and*

$$\mathrm{cap}_p(\{|b(x,u)| > k\}) \leq C \max\{\frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}}\}, \quad \forall k \geq 1. \tag{6.1.12}$$

for all $C = C(\|\mu\|_{\mathcal{M}_b(Q)}, \|b(x,u_0)\|_{L^1(\Omega)}, \|B\|_{L^1(\Omega)}, p)$.

Proof. See Section 6.4. □

We can now recall the approximation on diffuse measures, whose proof holds for any solution in the space $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, corresponding to the truncations of nonlinear potential of μ.

Proposition 6.4. *If $\mu \in \mathcal{M}_0(Q)$. Then, for every $\epsilon > 0$, there exists $\nu \in \mathcal{M}_0(Q)$ such that*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \epsilon \text{ and } \nu = w_t - \Delta_p w \text{ in } \mathcal{D}'(Q), \tag{6.1.13}$$

where $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Proof. See [**PPP2**], Theorem 1.1. □

We can also define the class of equi-diffuse measures, that will be play an essential role in the next.

Definition 6.5. A sequence of measures (μ_n) on Q is equi-diffuse, if for every $\eta > 0$ there exists $\delta > 0$ such that for every Borel measurable set $E \subset Q$

$$\mathrm{cap}_p(E) < \delta \implies |\mu_n|(E) < \eta \quad \forall n \geq 1. \tag{6.1.14}$$

Proposition 6.6. *If $\mu \in \mathcal{M}_0(Q)$ and ρ_n is a sequence of mollifiers on Q. Then the sequence $(\rho_n * \mu_n)$ is equi-diffuse.*

Proof. See [**PPP2**], Proposition 3.3. □

In order to deal with the renormalized formulation, we will often make use of the following auxiliary functions of real variable $\Theta_n(s) = T_1(s - T_n(s))$, $h_n(s) = 1 - \Theta_n(s)$, $S_n(s) = \int_0^s h_n(r)dr$, $\forall s \in \mathbb{R}$ (see Section 1.9) and another auxiliary functions that we will often use in the next sections; this functions can be introduced in terms of $T_k(s)$ and $S_k(s)$ and defined as follows,

$$S_{k,\sigma}(s) = \begin{cases} 0 & \text{if } |s| \geq k+\sigma \\ 1 & \text{if } |s| \leq k \\ \text{affine} & \text{otherwise} \end{cases} \quad h_{k,\sigma}(s) = 1 - S_{k,\sigma}(s) = \begin{cases} 1 & \text{if } |s| \geq k+\sigma \\ 0 & \text{if } |s| \leq k \\ \text{affine} & \text{otherwise} \end{cases} \quad T_{k,\sigma}(s) = \int_0^s S_{k,\sigma}(r)dr.$$

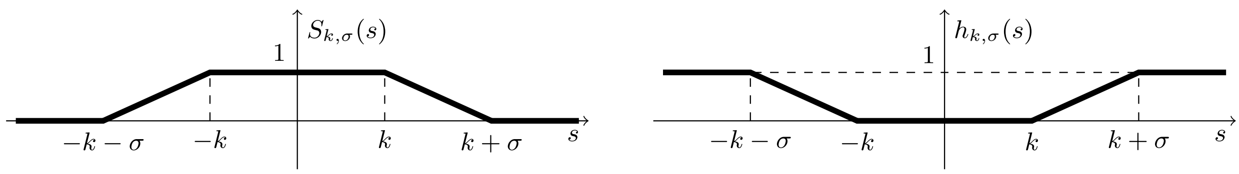


Figure 20. The functions $S_{k,\sigma}(s)$ and $h_{k,\sigma}(s)$

Notice that functions $T_{k,\sigma}(s)$ converges pointwise to $T_k(s)$ as σ goes to zero.

DEFINITION 6.7. A measurable function u is a renormalized solution of (6.1.1) if

$$v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \text{ for every } q < p - \frac{N}{N+1},$$

$$T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k > 0,$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le v<2n\}} a(t,x,u,\nabla u)\cdot\nabla v\psi\, dxdt = \int_Q \psi d\mu_s^+, \tag{6.1.15}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n<v\le n\}} a(t,x,u,\nabla u)\cdot\nabla v\psi\, dxdt = \int_Q \psi d\mu_s^-, \tag{6.1.16}$$

and the following equation holds

$$\begin{aligned} &-\int_\Omega S(b(x,u_0))\varphi(0)dx - \int_0^T \langle \varphi_t, S(v)\rangle dt + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla\varphi\, dxdt \\ &+\int_Q S''(v)a(t,x,u,\nabla u)\cdot\nabla v\varphi\, dxdt = \int_Q S'(v)\varphi d\tilde{\mu}_0, \end{aligned} \tag{6.1.17}$$

for any $S \in W^{2,\infty}(\mathbb{R})$ with S' has a compact support and $S(0)=0$, and for any $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\varphi(T,x)=0$.

REMARK 6.8. We first introduce some essential regularity results following the equation in the sense of distribution (6.1.17), notice that, thanks to our regularity assumptions and the choice of S, all terms in (6.1.17) are well defined since $T_k(b(x,u)-g)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$ and since S' has compact support. Indeed by taking M such that Supp $S' \subset]-M,M[$, since $S'(b(x,u)-g) = S''(b(x,u)-g) = 0$ as soon as $|b(x,u)-g| \ge M$, we can replace, everywhere in (6.1.17), $\nabla(b(x,u)-g)$ by $\nabla T_M(b(x,u)-g) \in (L^p(Q))^N$ and ∇u by $b_s(x,u)^{-1}(\nabla T_M(v) - \nabla_x b\chi_{\{|v|\le M\}} + \nabla g\chi_{\{|v|\le M\}}) \in (L^p(Q))^N$. Moreover, according to the assumptions $(6.1.3)-(6.1.4)$ and the definition of ∇u, $b_s(x,u)^{-1}(\nabla T_M(v) - (\nabla_x b(x,u) - \nabla g)\chi_{\{|v|\le M\}}) \in (L^p(Q))^N$, we have $\nabla(b(x,u)-g)$ is well defined and since $|u| \le \lambda^{-1}(M+|g|)$ as soon as $|v|\le M$ then, $|a(t,x,u,\nabla u)\chi_{\{|v|\le M\}}| \in (L^{p'}(Q))^N$.
We also have, for all S as above, $S(b(x,u)-g) = S(T_M(b(x,u)-g)) \in L^p(0,T;W_0^{1,p}(\Omega))$ and

$$\begin{aligned} &S'(b(x,u)-g)f \in L^1(Q);\\ &S'(b(x,u)-g)G \in (L^{p'}(Q))^N;\\ &S'(b(x,u)-g)a(t,x,u,\nabla u) \in (L^{p'}(Q))^N;\\ &S''(b(x,u)-g)a(t,x,u,\nabla u)\cdot\nabla T_M(b(x,u)-g) \in L^1(Q);\\ &S''(b(x,u)-g)G\cdot\nabla T_M(b(x,u)-g) \in L^1(Q). \end{aligned}$$

Thus, by equation (6.1.17), $(S(b(x,u)-g))_t$ belongs to the space $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, and therefore $S(b(x,u)-g)$ belongs to $C([0,T];L^1(\Omega))$ (see **[Po1]**, Theorem 1.1) and one can say that the initial datum is achieved in a weak sense, that is $S(b(x,u)-g)(0) = S(b(x,u)(0)-g(0) = S(b(x,u_0)$ in $L^1(\Omega)$ (recall that g has compact support in Q) for every renormalization S. Note also that, since $S(b(x,u)-g)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, we can use in (6.1.17) not only functions in $C_0^\infty(Q)$ but also in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

REMARK 6.9. The initial condition $S(v)(0) = S(b(x,u_0))$ is the renormalized version of the requirement that $v(0) = b(x,u_0)$. Observe also that conditions $(6.1.15)-(6.1.16)$ are equivalent to

$$\lim_{h\to\infty} \int_{\{h-1\le|v|\le h\}} a(t,x,u,\nabla u)\cdot\nabla u\xi\, dxdt = \int_Q \xi d\mu_s \tag{6.1.18}$$

for any $\xi \in C_c^\infty([0,T]\times\Omega)$.

Note that this formulation of renormalized solution does not depend on the decomposition of μ.

PROPOSITION 6.10. *Let u be a renormalized solution of* (6.1.1). *Then u satisfies Definition 6.7 for every decomposition* $(f, -\mathrm{div}(G), g)$ *of* μ.

Proof. See [**MR**], Proposition 2. □

Another definition of renormalized solutions for problem (6.1.1) can be stated as follows

Definition 6.11. A function $u \in L^1(Q)$ is a renormalized solutions of problem (6.1.1) if

$$b(x,u) - g \in L^\infty(0,T;L^1(\Omega)), \quad T_k(b(x,u)-g) \in L^p(0,T;W_0^{1,p}(\Omega)),$$

and if there exists a sequence of non-negative measure $\nu_k \in \mathcal{M}_b(Q)$ such that

$$\nu^k \to \mu_s \text{ tightly as } k \to +\infty, \tag{6.1.19}$$

and

$$\begin{aligned} &-\int_Q (T_k(b(x,u))\varphi_t dxdt + \int_Q a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(b(x,u)) - \nabla_x T_k(b(x,u)))) \cdot \nabla\varphi \, dxdt \\ &= \int_Q \varphi d\mu_0 + \int_Q \varphi d\nu^k + \int_\Omega T_k(b(x,u_0))\varphi(0)dx. \end{aligned} \tag{6.1.20}$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Remark 6.12. First observe that (6.1.20) implies that

$$T_k(b(x,u))_t - \operatorname{div}(a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(b(x,u)) - \nabla_x T_k(b(x,u)))$$

is a bounded measure, and since $T_k(b(x,u)) \in L^p(0,T;W_0^{1,p}(\Omega))$ this means that from the Lipschitz regularity in a and the fact that $|u|^{p-1} < |\frac{k}{\lambda}|^{p-1}$ that

$$T_k(b(x,u))_t - \operatorname{div}(a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(b(x,u)) - \nabla_x T_k(b(x,u)))) \in W' \cap \mathcal{M}_b(Q). \tag{6.1.21}$$

In addition, we have also the following equality for functions $T_k(b(x,u)-g)$

$$T_k(v) - \operatorname{div}(a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x,u))\chi_{\{|v|\le k\}}) = f - \operatorname{div}(G) + \nu_k \tag{6.1.22}$$

in Q. From Proposition 3.1 of [**PPP2**] the measure ν^k is diffuse, then we can recover from equation (6.1.20) the standard estimates known for nonlinear potentials. Moreover, if μ is diffuse the Definition 6.11 coincides with Definition 1 of [**MR**]. This fact is easy to check once we observe that non-negative measures that vanish tightly actually strongly converge to zero in $\mathcal{M}_b(Q)$.

It should be observed that we can consider a larger class of test functions.

Proposition 6.13. *Let u be a renormalized solution in the sense of Definition 6.11. Then we have*

$$\begin{aligned} &-\int_Q T_k(v)w_t \, dxdt + \int_Q a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x,u))\chi_{\{|v|\le k\}}) \cdot \nabla w \, dxdt \\ &= \int_Q f\tilde{w} \, dxdt + \int_Q G \cdot \nabla\tilde{w} \, dxdt + \int_Q \tilde{w} d\nu^k + \int_\Omega T_k(b(x,u_0))w(0)dx \end{aligned} \tag{6.1.23}$$

for every $\tilde{w} \in W \cap L^\infty(Q)$ such that $w(T) = 0$ (with $\tilde{w}$ being the unique cap_p quasi-continuous representative of w).

In Proposition 6.13 we essentially use test functions that depend on the solution itself and can be used also in (6.1.20). Note that renormalized solutions are distributional solutions of the same problem, see [**PPP2**], Proposition 4.2.

Proposition 6.14. *Let u be a renormalized solution of* (6.1.1)*. Then u satisfies, for every $k > 0$ and $\tau \le T$*

$$\int_\Omega \Theta_k(b(x,u))(\tau)dx + \int_0^\tau \int_\Omega b_s(x,u)a(t,x,u,\nabla u)\cdot\nabla u \, dxdt \le Ck(\|\mu\|_{\mathcal{M}_b(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}), \tag{6.1.24}$$

where $\Theta_k(s) = \int_0^s T_k(t)dt$.

Proof. See [**Pe3**], Proposition 3. □

Now we state the main result of this paper.

Theorem 6.15. *Under assumptions* (6.1.2)−(6.1.9)*, there exist at least a renormalized solution u of problem* (6.1.1) *in the sense of Definition 6.11.*

6.2. A priori estimates and main result

We start by proving a priori estimates and compactness arguments, defining a solutions as a limit of bounded sequence of suitable approximating problems. Moreover, we will postpone the proof of Theorem 6.15 at the next section, since it makes use of some techniques of Section 6.2. Now, let us come back to the fundamental decomposition theorem for general measure; as we said before, if $\mu \in \mathcal{M}_b(Q)$ one can split it in this way

$$\mu = \mu_0 + \mu_s = f - \operatorname{div}(G) + g + \mu_s^+ - \mu_s^-, \tag{6.2.1}$$

for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, and $\mu_s \perp \operatorname{cap}_p$, that is, μ_s is concentrated on a set $E \subset Q$ with $\operatorname{cap}_p(E) = 0$. There are many ways to approximate this measure looking for existence of solutions for problem (6.1.1); we will make the following choice; let

$$\mu_n = f_n - \operatorname{div}(G_n) + g_n + \lambda_n^{\oplus} - \lambda_n^{\ominus}, \tag{6.2.2}$$

where

$$\begin{cases} f_n \in C_0^\infty(Q) \text{ such that } f_n \rightharpoonup f \text{ weakly in } L^1(Q), \\ G_n \in C_0^\infty(Q) \text{ such that } G_n \to G \text{ strongly in } (L^{p'}(Q))^N, \\ g_n \in C_0^\infty(Q) \text{ such that } g_n \to g \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ \lambda_n^{\oplus} \in C_0^\infty(Q) \text{ such that } \lambda_n^{\oplus} \to \mu_s^+ \text{ in the narrow topology of measures}, \\ \lambda_n^{\ominus} \in C_0^\infty(Q) \text{ such that } \lambda_n^{\ominus} \to \mu_s^- \text{ in the narrow topology of measures}. \end{cases} \tag{6.2.3}$$

Moreover, let

$$u_0^n \in C_c^\infty(Q) \text{ such that } b(x,u_0^n) \to b(x,u_0) \text{ strongly in } L^1(\Omega). \tag{6.2.4}$$

Notice that this approximation can be easily obtained via a standard convolution argument. We also assume

$$\mu_n \in C_0^\infty(Q) \text{ such that } \|\mu_n\|_{L^1(Q)} \le C\|\mu\|_{\mathcal{M}_b(Q)} \text{ and } \|b(x,u_0^n)\|_{L^1(\Omega)} \le C\|b(x,u_0)\|_{L^1(\Omega)}. \tag{6.2.5}$$

Let us consider the following approximation of problem (6.1.1)

$$\begin{cases} (b(x,u_n))_t - \operatorname{div}(a(t,x,u_n,\nabla u_n)) = \mu_n & \text{in } (0,T)\times\Omega, \\ u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(x,u_n)(0) = b(x,u_0^n) & \text{a.e. in } \Omega. \end{cases} \tag{6.2.6}$$

Then from the well-known result of [**L**], there exist at least a weak solution $u_n \in C([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ of problem (6.2.6) such that $(b(x,u_n))_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and satisfies

$$\int_Q (b(x,u_n))_t \psi \, dxdt + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi \, dxdt = \int_Q \mu_n \psi \, dxdt \tag{6.2.7}$$

for any $\psi \in L^p(0,T;W_0^{1,p}(\Omega))$. Approximation (6.2.2) $-$ (6.2.5) yields standard compactness results [**DPP, MR, Pe1**] that we collect in the following Proposition.

PROPOSITION 6.16. *Let u_n and $v_n = b(x,u_n) - g_n$ defined as before. Then*

$$\begin{cases} \|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C, \quad \|v_n\|_{L^\infty(0,T;L^1(\Omega))} \le C, \\ \displaystyle\int_Q |\nabla T_k(u_n)|^p dxdt \le Ck, \quad \int_Q |\nabla T_k(v_n)|^p dxdt \le C(k+1). \end{cases} \tag{6.2.8}$$

Moreover, there exists a measurable functions u and $v = b(x,u) - g$ such that $T_k(u)$ and $T_k(v)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, u and v belongs to $L^\infty(0,T;L^1(\Omega))$, and up to a subsequence, for any $k > 0$, and for any $q < p - \frac{N}{N+1}$,

$$\begin{cases} u_n \to u \text{ a.e. in } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q), \\ v_n \to v \text{ a.e. in } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q), \\ T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q, \\ T_k(v_n) \rightharpoonup T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q. \end{cases} \tag{6.2.9}$$

Moreover u_n and v_n are bounded sequences in $L^\infty(0,T;W_0^{1,p}(\Omega))$; in particular, there exists u, v, σ_k and $\nabla_x b(x,u_n)$ such that (up to subsequences)

$$\begin{cases} \nabla u_n \to \nabla u \ \textit{a.e. in } Q, \\ \nabla v_n \to \nabla v \ \textit{a.e. in } Q, \\ a(t,x,u_n,\nabla u_n)\chi_{\{|v_n|\le k\}} \rightharpoonup \sigma_k \ \textit{weakly in } (L^{p'}(Q))^N, \\ \nabla_x b(x,u_n) \to \nabla_x b(x,u) \ \textit{strongly in } (L^p(Q))^N. \end{cases} \tag{6.2.10}$$

Proof. We choose $T_k(b(x,u_n))$ as a test function in (4.6) (the use of $T_k(b(x,u_n)-g_n)$ as a test function can be used to have estimates on v), we obtain

$$\begin{aligned} &\int_\Omega \Theta_k(b(x,u_n))(t)dx + \int_0^t\int_\Omega a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(b(x,u_n))\,dxdt \\ &= \int_0^t\int_\Omega T_k(b(x,u_n))d\mu_n + \int_\Omega \Theta_k(b(x,u_0^n))dx \end{aligned} \tag{6.2.11}$$

where we have set $t\in[0,T]$ and $\Theta_k(s)$ the primitive function of $T_k(s)$. It results from assumption (6.1.5) and the fact that $\|b(x,u_n)\|_{L^1(Q)}$ is bounded

$$\begin{aligned} &\int_\Omega \Theta_k(b(x,u_n))(t)dx + \int_{\{|b(x,u_n)|\le k\}} b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt \\ &+ \int_{\{|b(x,u_n)|\le k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla_x b(x,u_n)dxdt \le k\|\mu\|_{\mathcal{M}_b(Q)} + \int_\Omega \Theta_k(b(x,u_0^n))dx. \end{aligned}$$

Then

$$\begin{aligned} &\int_\Omega \Theta_k(b(x,u_n))(t) + \alpha\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt \le k\|\mu\|_{\mathcal{M}_b(Q)} + \beta\int_{E_k} L(t,x)\cdot|\nabla_x b(x,u_n)| \\ &+ \beta\int_{E_k}|u_n|^{p-1}\cdot|\nabla_x b(x,u_n)|dxdt + \frac{\beta}{\lambda}\int_{E_k} b_s(x,u_n)|\nabla u_n|^{p-1}\cdot|\nabla_x b(x,u_n)| + k\|b(x,u_0^n)\|_{L^1(\Omega)}, \end{aligned}$$

where $E_k=\{(t,x):|b(x,u_n)|\le k\}$, using (6.1.3) and by means of Young's inequality, we obtain

$$\begin{aligned} \beta\int_{E_k}|\nabla u_n|^{p-1}\cdot|\nabla_x b(x,u_n)|dxdt &\le \frac{\beta}{\lambda}\int_{E_k} b_s(x,u_n)|\nabla u_n|^{p-1}\cdot|\nabla_x b(x,u_n)|dxdt \\ &\le \frac{\alpha}{2}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt + \frac{T}{p}(\Lambda+1)(\frac{2\beta p'}{\alpha\lambda})^{p-1}\|B\|^p_{L^p(\Omega)}, \end{aligned}$$

and

$$\int_{E_k}|u_n|^{p-1}\cdot|\nabla_x b(x,u_n)|dxdt \le \int_{E_k}|\frac{k}{\lambda}|^{p-1}|\nabla_x b(x,u_n)|dxdt \le C\|B\|^p_{L^p(\Omega)},$$

since $\Theta_k(s)\ge 0$ and $|\Theta_1(s)|\ge |s|-1$, we get

$$\begin{aligned} &\int_\Omega |b(x,u_n)(t)|dx + \frac{\alpha}{2}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt \\ &\le k(\|\mu\|_{\mathcal{M}_b(Q)} + \|b(x,u_0^n)\|_{L^1(\Omega)}) + C(\|L\|^{p'}_{L^{p'}(Q)} + \|B\|^p_{L^p(\Omega)}). \end{aligned}$$

Finally, we get

$$\int_\Omega |b(x,u_n)(t)|dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla T_k(b(x,u_n))|^p dxdt \le C(k+1) \quad \forall k>0,\ \forall t\in[0,T].$$

From the previous estimates, we deduce that

$$\|b(x,u_n)\|_{L^\infty(0,T;L^1(\Omega))}\le C \text{ and } \int_Q |\nabla T_k(b(x,u_n))|^p dxdt \le C(k+1).$$

Similarly we can get the estimate on $v_n = b(x,u_n) - g_n$ if we choose $T_k(v_n)$ as test function in (6.2.6).

$$
\begin{aligned}
&\int_\Omega \Theta_k(v_n)(t)dx + \alpha \int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt \\
&\le \int_\Omega \Theta_k(b(x,u_0^n))dx + k\|f\|_{L^1(Q)} + \int_{E_k} |G\cdot \nabla T_k(v_n)|dxdt \\
&+ \beta(\int_{E_k} L(t,x)|\nabla g_n|dxdt + \int_{E_k} |u_n|^{p-1}|\nabla g_n|dxdt + \int_{E_k} |\nabla u_n|^{p-1}dxdt) \\
&+ \int_{E_k} |a(t,x,u,\nabla u_n)\cdot \nabla_x b(x,u_n)|dxdt + \int_Q T_k(v_n)d\lambda_n^\oplus - \int_Q T_k(v_n)d\lambda_n^\ominus,
\end{aligned}
$$

where C is a constant independent on n and $E_k = \{(t,x) : |b(x,u_n) - g_n| \le k\}$. Using (6.1.3) and by means of Young's inequality, we have

$$
\begin{aligned}
&\int_{E_k} |G\cdot \nabla T_k(v_n)|dxdt \le \frac{\alpha}{2p}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt + C(\|B\|^p_{L^p(\Omega)} + \|G_n\|^{p'}_{L^{p'}()Q} + \|\nabla g_n\|^p_{L^p(Q)}), \\
&\int_{E_k} |u_n|^{p-1}|\nabla g_n|dxdt \le \int_{E_k} (k+|g_n|)^{p-1}|\nabla g|dxdt \le C(\|g_n\|^p_{L^p(Q)} + \|\nabla g_n\|^p_{L^p(Q)}), \\
&\int_{E_k} |a(t,x,u_n,\nabla u_n)\nabla_x b(x,u_n)|dxdt \le \frac{\alpha}{4p'}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt + \frac{T}{p}(\Lambda+1)(\frac{4\beta}{\alpha\lambda})^{p-1}\|B\|^p_{L^p(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta\int_{E_k} |\nabla u_n|^{p-1}|\nabla g_n|dxdt &\le \frac{\beta}{\lambda}\int_{E_k} b_s(x,u_n)|\nabla u_n|^{p-1}|\nabla g_n|^p dxdt \\
&\le \frac{\alpha}{4p'}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt + \frac{1}{p}(\Lambda+1)(\frac{4\beta}{\alpha\lambda})^{p-1}\int_{E_k} |\nabla g_n|^p dxdt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_\Omega \Theta_k(v_n)(t)dx + \frac{\alpha}{2}\int_{E_k} b_s(x,u_n)|\nabla u_n|^p dxdt \\
&\le C(\|L\|^{p'}_{L^{p'}(Q)} + \|g_n\|^p_{L^p(Q)} + \|\nabla g_n\|^p_{L^p(Q)} + \|B\|^p_{L^p(\Omega)} + \|G_n\|^{p'}_{L^{p'}(Q)}) \\
&+ k(\|f_n\|_{L^1(Q)} + \int_Q d\lambda_n^\oplus - \int_Q d\lambda_n^\ominus) + \int_\Omega \Theta_k(b(x,u_0^n))dx.
\end{aligned}
$$

G_n is bounded in $L^{p'}(Q)$, g_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f_n, $\lambda_n^\oplus$, $\lambda_n^\ominus$ are bounded in $L^1(Q)$ and $b_n(x,u_0^n)$ is bounded in $L^1(\Omega)$, we obtain

$$
\int_\Omega \Theta_1(v_n)(t)dx \le C \quad \forall t\in[0,T],
$$

which implies the estimate of v_n in $L^\infty(0,T;L^1(\Omega))$, and also

$$
\int_Q |\nabla u_n|^p \chi_{\{|v_n|\le k\}}dxdt \le C(k+1),
$$

which yields that $T_k(v_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k>0$ (recall that g_n is itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). □

6.3. Proof of the main result

In this section we prove Theorem 6.15. From here on ω will indicate any quantity that vanishes as the parameters in its argument go to their limit point with the same order in which they appear, that is, as an example $\lim_{\delta\to 0^+}\limsup_{m\to+\infty}\limsup_{n\to\infty}|\omega(n,m,\delta)| = 0$. Moreover, for the sake of simplicity, in what follows, the convergences, even if not explicitly stressed, may be understood to be taken possibly up to a suitable subsequence extraction. The proof is divided into *3 Steps*. In *Step. 1*, we establish a estimate on the energy term in $L^1(Q)$. In *Step. 2*, the limit ν_n^k is proved to converge to μ_s and that (6.1.19) holds. In *Step. 3*, we define cut-off functions which allows us to control the singular term of measure when passing to the limit. In the next, we suppose the following assumption, which is the key point to assure the weak convergence of $a(t,x,u_n,\nabla u_n)$ to $a(t,x,u,\nabla u)$ in $L^{p'}(Q)$, especially in the zone $\{|v_n|\leq k)$ (see equation (6.3.12))

$$|a(t,x,u,\nabla u)|\leq C(u)+|\nabla u|^{p-1}$$

where C is a bounded continuous functions in $\mathbb{R}$.

Step 1. Basic estimates in $L^1(Q)$. Here and elsewhere in the paper, $H_{k,\sigma}(s)$ will be the primitive function of $h_{k,\sigma}(s)$ and for technical reasons, we use the special test functions constructed from the function $S_k(s)$ and $T_k(s)$ defined in Section 6.1. We take $h_{k,\sigma}(b(x,u_n))$ as test function in the weak formulation of (6.2.6). Then

$$\begin{aligned}&\int_\Omega H_{k,\sigma}(b(x,u_n)(T))dx+\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla h_{k,\sigma}(b(x,u_n))dxdt\\&=\int_Q \mu_n h_{k,\sigma}(b(x,u_n))dxdt+\int_\Omega H_{k,\sigma}(b(x,u_0^n))dx.\end{aligned}\tag{6.3.1}$$

So that

$$\begin{aligned}&\int_\Omega H_{k,\sigma}(b(x,u_n))(T)dx+\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt\\&\qquad+\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}a(t,x,u_n,\nabla u_n)\nabla_x b(x,u_n)dxdt\\&\qquad=\int_Q h_{k,\sigma}(b(x,u_n))d\mu_n+\int_\Omega H_{k,\sigma}(b(x,u_0^n))dx,\end{aligned}\tag{6.3.2}$$

so that, dropping positive terms and using the fact that

$$\begin{aligned}&\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt\\&\leq C\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}(|k|^p+|L(t,x)|^{p'}+|B|^p)dxdt+\int_{\{|b(x,u_n)|>k\}}d\mu_n+\int_{\{|b(x,u_0^n)|>k\}}|b(x,u_0^n)|dx,\end{aligned}$$

then

$$\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt\leq C(k)+\int_{\{|b(x,u_n)|>k\}}d\mu_n+\int_{\{|b(x,u_0^n)|>k\}}|b(x,u_0^n)|dx,$$

we readily have the following estimate on the energy term

$$\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\ dxdt\leq C(k,n),\tag{6.3.3}$$

so that, there exist a constant $C=\frac{C(k,n)}{\lambda}$ such that

$$\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\ dxdt\leq C,\tag{6.3.4}$$

because of this fact, there exist a bounded Radon measure λ_k^n such that, as σ goes to zero

$$\frac{1}{\sigma}a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\chi_{\{k\leq|b(x,u_n)|<k+\sigma\}}\rightharpoonup\lambda_k^n\text{ weakly–}*\text{ in }\mathcal{M}(Q).\tag{6.3.5}$$

Now, looking at the equation in (6.2.6) and using $S_{k,\sigma}(b(x,u_n))\varphi$ as test function with $\varphi\in C_c^\infty(Q)$,

$$\int_0^T\langle b(x,u_n)_t,S_{k,\sigma}(b(x,u_n))\varphi\rangle dt+\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla(S_{k,\sigma}(b(x,u_n))\varphi)dxdt=\int_Q S_{k,\sigma}(b(x,u_n))\varphi d\mu_n$$

Then,

$$\begin{aligned}&\int_0^T\langle b(x,u_n)_t,S_{k,\sigma}(b(x,u_n))\varphi\rangle dt-\frac{1}{\sigma}\int_{\{k\leq|b(x,u_n)|<k+\sigma\}}a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)\varphi\ dxdt\\&+\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\varphi S_{k,\sigma}(b(x,u_n))dxdt=\int_Q S_{k,\sigma}(b(x,u_n))\varphi d\mu_n\end{aligned}\tag{6.3.6}$$

as σ tends to infinity, we have

$$\int_0^T \langle T_k(b(x,u_n))_t, \varphi\rangle dt - \int_Q \varphi d\lambda_k^n + \int_{\{|b(x,u_n)|\le k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla\varphi\ dxdt = \int_{\{|b(x,u_n)|\le k\}} \varphi d\mu_n.$$

From the definition of $T_k(s)$, we have

$$\begin{aligned}&\int_0^T \langle T_k(b(x,u_n))_t, \varphi\rangle dt + \int_Q a(t,x,u_n,(b_s(x,u_n))^{-1}(\nabla T_k(b(x,u_n)-\nabla_x b(x,u_n)))\cdot\nabla\varphi\ dxdt\\ &= \int_Q \varphi d\lambda_k^n + \int_Q \varphi d\mu_0^n - \int_{\{|b(x,u_n)|\ge k\}} \varphi d\mu_0^n + \int_{\{|b(x,u_n)|\le k\}} \varphi d\mu_s^n,\end{aligned} \tag{6.3.7}$$

then

$$\begin{aligned}&T_k(b(x,u_n))_t - \operatorname{div}(a(t,x,u_n,(b_s(x,u_n))^{-1}(\nabla T_k(b(x,u_n)) - \nabla_x b(x,u_n))) - \mu_0^n\\ &= \lambda_k^n - \mu_0^n\chi_{\{|b(x,u_n)|\ge k\}} + \mu_s^n\chi_{\{|b(x,u_n)|\le k\}}\end{aligned} \tag{6.3.8}$$

in the sense of distributions. We define the measure ν_n^k as

$$\nu_n^k = \lambda_k^n - \mu_0^n\chi_{\{|b(x,u_n)|\ge k\}} + \mu_s^n\chi_{\{|b(x,u_n)|\le k\}} \tag{6.3.9}$$

we have that ν_n^k is bounded in $L^1(Q)$ and so there exist $\nu^k\in\mathcal{M}(Q)$ such that

$$\nu_n^k \rightharpoonup \nu^k \text{ weakly–* in } \mathcal{M}(Q). \tag{6.3.10}$$

Then, by convergence arguments of Proposition 6.16, we have that

$$T_k(b(x,u))_t - \operatorname{div}(a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(b(x,u)) - \nabla_x b(x,u)) = \mu_0^n + \nu^k \text{ in } \mathcal{D}'(Q). \tag{6.3.11}$$

Therefore, thanks to distributional formulation (6.3.11), we have for every $\varphi\in C_c^\infty(Q)$

$$\begin{aligned}&\int_0^T \langle (b(x,u_n) - T_k(b(x,u_n)))_t, \varphi\rangle dt\\ &+ \int_Q (a(t,x,u_n,\nabla u_n) - a(t,x,u,(b_s(x,u))^{-1}(\nabla T_k(b(x,u)) - \nabla_x b(x,u)))\cdot\nabla\varphi\ dxdt\\ &= \int_Q \varphi d(\mu_0^n - \mu_0) + \int_Q \varphi d(\mu_s^n - \nu^k) + \int_\Omega (b(x,u_0^n) - T_k(b(x,u_0)))\varphi\ dx\end{aligned} \tag{6.3.12}$$

and, passing to the limit as n tends to infinity, we obtain

$$\nu^k \to \mu_s \text{ in the sense of distributions as } n \text{ tends to } +\infty. \tag{6.3.13}$$

Step 2. Near and far from E. We will use also the following result

LEMMA 6.17. *Let μ_s be a non-negative bounded Radon measure on Q, concentrated on a set E of zero $p-$capacity. Then, for every $\delta>0$, there exists a compact set $K_\delta\subseteq E$ with*

$$\mu_s(E\backslash K_\delta)\le\delta$$

and there exist $\psi_\delta\in C_0^1(Q)$ such that

$$\psi_\delta\equiv 1 \text{ on } K_\delta \text{ and } 0\le\psi_\delta\le 1.$$

Moreover,

$$\|\psi_\delta\|_S\le\delta \text{ and } \int_Q (1-\psi_\delta)d\mu_s = \omega(\delta).$$

PROOF. See [**Pe1**], Lemma 5. □

Now, let ψ_δ as in Lemma 6.17, let us mention that the use of these type of cut-off functions is to deal with, separately, the regular and the singular parts of the data and they first introduced in [**DMOP**] in the elliptic framework and then used in [**Pe1**] to deal with parabolic problems. Then we have

$$\int_Q \varphi d\nu^k = \int_Q \varphi\psi_\delta d\nu^k + \int_Q \varphi(1-\psi_\delta)d\nu^k$$

where ψ_δ is chosen as in Lemma 6.17. Then we want to show that

$$\int_Q \varphi\psi_\delta d\nu^k = \omega(k,\delta), \tag{6.3.14}$$

and

$$\int_Q \varphi(1-\psi_\delta) d\nu^k = \omega(k,\delta). \tag{6.3.15}$$

We will prove the estimate (6.3.14) near E and the estimate (6.3.15) far from E, alternatively.

Step 3. Near E. Thanks to the result (6.3.13) we have that

$$\int_Q \varphi\psi_\delta d\nu^k \to \int_Q \varphi\psi_\delta d\mu_s \quad \text{as } k\to+\infty. \tag{6.3.16}$$

Recalling that $\psi_\delta \equiv 1$ on K_δ, we have

$$\begin{aligned}\int_Q \varphi\psi_\delta d\mu_s = \int_E \varphi\psi_\delta d\mu_s &= \int_{\{E\setminus K_\delta\}} \varphi\psi_\delta d\mu_s + \int_{K_\delta} \varphi d\mu_s \\ &\le \|\varphi\|_{L^\infty(Q)}\mu_s(E\setminus K_\delta) + \int_Q \varphi d\mu_s \\ &\le \delta\|\varphi\|_{L^\infty(Q)} + \int_Q \varphi d\mu_s.\end{aligned}$$

Then

$$\int_Q \varphi\psi_\delta d\mu_s \to \int_Q \varphi d\mu_s \quad \text{as } \delta\to 0. \tag{6.3.17}$$

Putting together (6.3.16) and (6.3.17), we have

$$\int_Q \varphi\psi_\delta d\nu^k \to \int_Q \varphi d\mu_s \quad \text{as } k\to+\infty \text{ and } \delta\to 0. \tag{6.3.18}$$

Step 4. The estimate far from E. Let us analyse the term (6.3.15). Using the definition of ν^k we have

$$\begin{aligned}\int_Q \varphi(1-\psi_\delta)d\nu^k = \lim_{n\to\infty}\Big[\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_{\{k\le|b(x,u_n)|<k+\sigma\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n(1-\psi_\delta)\varphi\ dxdt \\ + \int_{\{|b(x,u_n)|\le k\}}\varphi(1-\psi_\delta)d\mu_s^n - \int_{\{|b(x,u_n)|>k\}}\varphi(1-\psi_\delta)d\mu_0^n\Big],\end{aligned} \tag{6.3.19}$$

by means of Lemma 6.3 and Lemma 6.17, we readily have

$$\int_{\{|b(x,u_n)|>k\}}\varphi(1-\psi_\delta)d\mu_0^n \le \|\varphi\|_{L^\infty(Q)}|\mu_0^n(\{|b(x,u_n)|>k\})| \le \omega(n,k) \tag{6.3.20}$$

and, again by Lemma 6.17 we get

$$\int_{\{|b(x,u_n)|>k\}}\varphi(1-\psi_\delta)d\mu_s^n \le \|\varphi\|_{L^\infty(Q)}\int_Q(1-\psi_\delta)d\mu_s \le \omega(n,\delta). \tag{6.3.21}$$

We need the following argument similar to the one obtained in [**Pe1**], and the proof will be done with the aid of test functions depending on ψ_δ.

LEMMA 6.18. *There exist a constant $C=\omega(\sigma,n,k,\delta)>0$ such that for every $k>0$*

$$\frac{1}{\sigma}\int_{\{k<|b(x,u_n)|\le k+\sigma\}} a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)\varphi(1-\psi_\delta)dxdt \le C. \tag{6.3.22}$$

PROOF. We choose $h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)$ as a test function in (6.2.6) where $h_{k,\sigma}(s)$ appears in Figure 1. Thus

$$\begin{aligned}&\int_0^T\langle b(x,u_n)_t, h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)\rangle dt + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)dxdt \\ &= \int_Q h_{k,\sigma}(b(x,u_n))d\mu_n.\end{aligned} \tag{6.3.23}$$

Then

$$\begin{aligned}
&\int_0^T \langle H_{k,\sigma}(b(x,u_n))_t, (1-\psi_\delta)\rangle dt \\
&- \frac{1}{\sigma}\int_{\{k\le b(x,u_n)<k+\sigma\}} a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)(1-\psi_\delta)dxdt \\
&- \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_\delta h_{k,\sigma}(b(x,u_n))dxdt \\
&= \int_Q h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)d\mu_0^n + \int_Q h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)d\mu_s^n,
\end{aligned} \tag{6.3.24}$$

and easily

$$\begin{aligned}
&\int_\Omega H_{k,\sigma}(b(x,u_n))(T)(1-\psi_\delta(T))dx && (A)\\
&- \int_\Omega H_{k,\sigma}(b(x,u_n))(0)(1-\psi_\delta(0))dx && (B)\\
&- \int_\Omega H_{k,\sigma}(b(x,u_n))(\psi_\delta)_t dxdt && (C)\\
&- \frac{1}{\sigma}\int_{\{k\le |b(x,u_n)|<k+\sigma\}} a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)(1-\psi_\delta)dxdt && (D)\\
&- \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_\delta h_{k,\sigma}(b(x,u_n))dxdt && (E)\\
&\le \int_{\{|b(x,u_n)|>k\}} h_{k,\delta}(b(x,u_n))(1-\psi_\delta)d\mu_0^n && (F)\\
&+ \int_Q h_{k,\sigma}(b(x,u_n))(1-\psi_\delta)d\mu_s^n. && (G)
\end{aligned}$$

Using the convergence in $L^1(Q)$ of $b(x,u_n)$, we have

$$(C) = \omega(n,k), \tag{6.3.25}$$

while

$$|(F)| + |(G)| \le \int_{\{|b(x,u_n)|>k\}} (1-\psi_\delta)d\mu_0^n + \int_Q (1-\psi_\delta)d\mu_s^n = \omega(n,k). \tag{6.3.26}$$

On the other hand, using the regularity of ψ_δ and $|a(t,x,u_n,\nabla u_n)|$, observing that ψ_δ goes to zero in Q, we have

$$(E) = \omega(n,k). \tag{6.3.27}$$

Using the condition $b(x,0)=0$, we have that the second term in the left hand side are $\omega(n,\delta)$. So that from all these facts we get

$$\lim_{n\to\infty}[\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_{\{k\le|b(x,u_n)|<k+\sigma\}} a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)(1-\psi_\delta)dxdt] = \omega(k,\delta). \tag{6.3.28}$$

□

Finally, from (6.3.19) and Lemma 6.18, we conclude

$$\int_Q \varphi(1-\psi_\delta)d\nu^k = \omega(k,\delta), \tag{6.3.29}$$

for all $\varphi \in C^1(\overline{Q})$. Then from (6.3.18) and (6.3.29) and by a density result, we have for all $\varphi \in C(\overline{Q})$,

$$\nu^k \to \mu_s \text{ tightly (in measure) as } k \text{ tends to infinity.}$$

6.4. Some further properties and remarks

As we have seen, the goal of this approach will be to pass to the limit using the equation solved by the truncations of u_n, see Definition 6.11. The major advantage of this approach is that we can perform the passage to the limit without using the strong convergence of the truncations in $L^p(0,T;W_0^{1,p}(\Omega))$ and the proof is based on the properties of the truncations of the renormalized solutions. Let us complete this approach by proving an asymptotic reconstruction property of the singular part of the measure.

PROPOSITION 6.19. *Let u_n be solution of* (6.2.6)*, then*

$$\lim_{h\to\infty} \limsup_{n\to\infty} \int_{\{h-1\le|b(x,u_n)|\le h\}} b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\psi\ dxdt = \int_Q \psi d\mu_s \tag{6.4.1}$$

for every $\psi\in C_c^\infty([0,T]\times\Omega)$.

In the next, we prove the following Lemma, which is the key point to control singular sets where μ is concentrated and that will be developed in the proof of Proposition 6.19.

LEMMA 6.20. *Let u_n be solution of* (6.2.6)*, $k>0$ and let ψ_δ be as in Lemma 6.17. Then*

$$\int_Q \mu_s^n(k-|b(x,u_n)|)^+\psi_\delta = \omega(n,\delta) \tag{6.4.2}$$

PROOF. We multiply the equation (6.2.6) by $(k-|b(x,u_n)|)^+\psi_\delta$ where ψ_δ is given by Lemma 6.17 and we integrate over Q, we get

$$\begin{aligned}
&-\int_Q\Big(\int_0^{|b(x,u_n)|}(k-s)^+ds\Big)(\psi_\delta)_t dxdt + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_\delta(k-|b(x,u_n)|)^+dxdt\\
&=\int_Q a(t,x,u_n,(b_s(x,u_n))^{-1}(\nabla T_k(b(x,u_n))-\nabla_x b(x,u_n))\cdot\nabla T_k(b(x,u_n))\psi_\delta dxdt\\
&+\int_Q(k-|b(x,u_n)|)^+\psi_\delta d\mu_n + \int_\Omega\Big(\int_0^{|b(x,u_0^n)|}(k-s)^+ds\Big)\psi_\delta(0)dx\\
&+\int_Q(k-|b(x,u_n)|)^+\psi_\delta d\mu_n + \int_\Omega\Big(\int_0^{|b(x,u_0^n)|}(k-s)^+ds\Big)\psi_\delta(0)dx
\end{aligned} \tag{6.4.3}$$

Now, using Proposition 6.16, observing that $\int_0^{|b(x,u_n)|}(k-s)^+ds\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and that ψ_δ goes to zero in S, we get both

$$-\int_Q\Big(\int_0^{|b(x,u_n)|}(k-s)^+ds\Big)(\psi_\delta)_t = \omega(n,\delta), \tag{6.4.4}$$

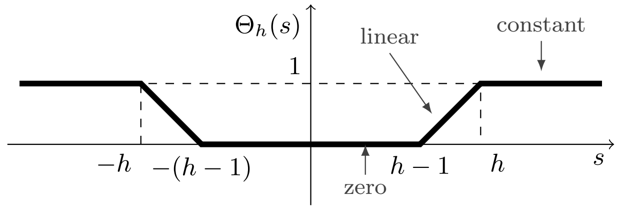
$$\begin{aligned}
&\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_\delta(k-|b(x,u_n)|)^+dxdt\\
&=\int_Q a(t,x,u_n,(b_s(x,u_n))^{-1}(\nabla T_k(b(x,u_n))-\nabla_x b(x,u_n))\cdot\nabla\psi_\delta(k-|b(x,u_n)|)^+ = \omega(n,\delta).
\end{aligned} \tag{6.4.5}$$

So that, dropping nonnegative terms in the right-hand side, we deduce (6.4.2). Let us also observe that, as a by-product, we also have the following property of the energy of the truncations near the singular set

$$\alpha\lambda\int_Q|\nabla u_n|^p dxdt \le \int_Q b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\psi_\delta dxdt \le \omega(n,\delta). \tag{6.4.6}$$

□

Proof of Proposition 6.19. Let



$$\Theta_h(s)=\begin{cases}1 & \text{if } |s|\geq h,\\ 0 & \text{if } |s|<h-1,\\ \text{affine} & \text{otherwise,}\end{cases}$$

FIGURE 21. The function $\Theta_h(s)$

and let us take $\Theta_h(b(x,u_n))\Psi$ as test function in (6.2.6), where $\Psi \in C_c^\infty(Q)$, to have

$$\begin{aligned}&-\int_Q\Big(\int_0^{|b(x,u_n)|}\Theta_h(s)ds\Big)\Psi_t dxdt\\ &+\int_{\{h-1\leq|b(x,u_n)|<h\}} a(t,x,u_n,\nabla u_n)\cdot\nabla b(x,u_n)\Psi dxdt\\ &+\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\Psi\Theta_h(b(x,u_n))dxdt\\ &=\int_Q \Psi\Theta_h(b(x,u_n))d\mu_0^n+\int_Q\Psi\Theta_h(b(x,u_n))d\mu_s^n\\ &+\int_\Omega\Big(\int_0^{|b(x,u_0^n)|}\Theta(s)ds\Psi(0)dx\end{aligned}\tag{6.4.7}$$

Let us analyse the previous terms one by one. First of all, thanks to Proposition 6.16 we easily get

$$\begin{aligned}&-\int_Q\Big(\int_0^{|b(x,u_n)|}\Theta_h(s)ds\Big)\Psi_t dxdt=\omega(n,k),\\ &\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\Psi\Theta_h(b(x,u_n))dxdt=\omega(n,k).\end{aligned}\tag{6.4.8}$$

Similarly dropping the term at $t=0$ and using the fact that $|\Theta_h(s)-1|\leq(h-s)^+$ and Lemma 6.20 we have

$$\begin{aligned}&\int_Q\Psi\Theta_h(b(x,u_n))d\mu_0^n+\int_Q\Psi\Theta_h(b(x,u_n))d\mu_s^n\\ &\leq\Big|\int_{\{|b(x,u_n)|\geq h-1\}}\Psi d\mu_0^n\Big|+\Big|\int_Q\Psi d\mu_s^n\Big|+\Big|\int_Q\Psi(\Theta_h(b(x,u_n)-1)d\mu_s^n\Big|\\ &\leq\|\Psi\|_{L^\infty(Q)}\Big(\Big|\int_{\{|b(x,u_n)|\geq h-1\}}d\mu_0^n\Big|+\Big|\int_Q d\mu_s^n\Big|+\Big|\int_Q(h-b(x,u_n))^+\psi_\delta d\mu_s^n\Big|+\Big|\int_Q(1-\psi_\delta)d\mu_s^n\Big|\Big).\\ &\leq\omega(n,k)+\omega(n,\delta)=\omega(n,k,\delta).\end{aligned}\tag{6.4.9}$$

Finally, gathering together all these results we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h-1\leq|b(x,u_n)|<h\}}b_s(x,u_n)a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\Psi dxdt=\int_Q\Psi d\mu_s.$$

Proof of Lemma 6.3. Let us now consider the capacitary estimate of renormalized solutions: we want to prove that u satisfies (6.1.12) in Lemma 6.3, we still use the notations introduced in Section 1.9 and Section 6.1, in particular, for simplicity we consider the case of $\tilde a(t,x,\zeta)=a(t,x,u(t,x),\zeta)=|\nabla\zeta|^{p-2}\zeta$ (i.e., $p-$Laplacian operator), so that $\tilde L=L+|u|^{p-1}$, then the function $\tilde a$ satisfies

$$|\tilde a(t,x,\zeta)\leq\beta(\tilde L+|\zeta|^{p-1})\text{ for a.e. }(t,x)\in Q\text{ and all }\zeta\in\mathbb{R}^N,$$

and (6.1.5), (6.1.6) and (6.1.7) (without the dependence in s). Hence, the problem (6.1.1) becomes

$$\begin{cases}b(x,\tilde u)_t-\operatorname{div}(\tilde a(t,x,\nabla\tilde u))=\mu & \text{in }(0,T)\times\Omega,\\ \tilde u=0 & \text{in }(0,T)\times\partial\Omega,\\ b(x,\tilde u)(0)=b(x,u_0) & \text{in }\Omega,\end{cases}$$

and we consider also the condition $p > \frac{2N+1}{N+1}$, we assume in addition that $\mu \in \mathcal{M}_b(Q)$ and $b(x,u_0) \in L^1(\Omega)$, hence, we have $b(x,\tilde{u}) \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Actually, the proof will be split into three parts, in the first one we obtain the basic estimates.

Step. 1 Estimates on $T_k(b(x,\tilde{u}))$ in the space $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. For every $\tau \in \mathbb{R}$, let

$$\Theta_k(\tau) = \int_0^s T_k(\sigma)d\sigma.$$

Take $r \in [0,T]$. Applying (6.1.1) with $v = T_k(b(x,\tilde{u}))$ and $\psi = \Theta_k$, $s = 0$ and $t = r$, we have

$$\int_\Omega \Theta_k(b(x,\tilde{u}))(r)dx + \int_0^r \int_\Omega \tilde{a}(t,x,\nabla\tilde{u}) \cdot \nabla T_k(b(x,\tilde{u}))dxdt \le k\|\mu\|_{\mathcal{M}_b(Q)} + \int_\Omega \Theta_k(b(x,u_0))dx,$$

Observing that $\frac{T_k(s)^2}{2} \le \Theta_k(s) \le k|s|$, $\forall s \in \mathbb{R}$, we have

$$\begin{aligned}&\int_\Omega \frac{[T_k(b(x,\tilde{u}))(r)]^2}{2}dx + \int_0^r \int_\Omega \tilde{a}(t,x,\nabla\tilde{u}) \cdot \nabla_x b(x,\tilde{u})\chi_{\{|b(x,\tilde{u})|\le k\}}dxdt\\ &+ \int_0^r \int_\Omega b_s(x,\tilde{u})\tilde{a}(t,x,\nabla\tilde{u}) \cdot \nabla\tilde{u}\chi_{\{|b(x,\tilde{u}|\le k\}} \le k(\|\mu\|_{\mathcal{M}_b(Q)} + \|b(x,u_0)\|_{L^1(\Omega)})\end{aligned}$$

for ay $r \in [0,T]$. In particular we deduce

$$\begin{aligned}&\int_\Omega \frac{[T_k(b(x,\tilde{u}))(t)]^2}{2}dx + \alpha\int_{\{|b(x,\tilde{u})|\le k\}} b_s(x,\tilde{u})|\nabla\tilde{u}|^p dxdt\\ &\le kM + \frac{\alpha}{2}\int_{\{|b(x,\tilde{u})|\le k\}} b_s(x,\tilde{u})|\nabla\tilde{u}|^p dxdt + \frac{T}{p}(\Lambda+1)(\frac{2\beta p'}{\alpha\lambda})^{p-1}\|B\|^p_{L^p(\Omega)}\end{aligned} \tag{6.4.10}$$

and then we have,

$$\int_\Omega \frac{[T_k(b(x,\tilde{u}))(t)]^2}{2}dx + \frac{\alpha}{2}\int_{\{|b(x,\tilde{u})|\le k\}} b_s(x,\tilde{u})|\nabla\tilde{u}|^p dxdt \le kM + C\|B\|^p_{L^p(\Omega)}.$$

Then

$$\|T_k(b(x,\tilde{u}))\|^2_{L^\infty(0,T;L^2(\Omega))} \le CkM \text{ and } \|T_k(b(x,\tilde{u}))\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le CkM, \tag{6.4.11}$$

where

$$M = \|\mu\|_{\mathcal{M}_b(Q)} + \|b(x,u_0)\|_{L^2(\Omega)} + \|B\|^p_{L^p(\Omega)}. \tag{6.4.12}$$

Step. 2 Estimates in W. In order to deduce some estimates in W, we use an idea from **[P]**. By standard results there exists a unique solution $z \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ of the backward problem

$$\begin{cases}-z_t - \Delta_p z = -2\Delta_p T_k(b(x,\tilde{u})) & \text{in } (0,T)\times\Omega,\\ z = 0 & \text{on } (0,T)\times\partial\Omega,\\ z = T_k(b(x,\tilde{u})) & \text{on } \{T\}\times\Omega.\end{cases} \tag{6.4.13}$$

Let us multiply (6.4.13) by z and integrate between z and T. Using Young's inequality, we obtain

$$\int_\Omega \frac{[z(\tau)]^2}{2}dx + \frac{1}{2}\int_0^T \int_\Omega b_s(x,\tilde{u})|\nabla z|^p dxdt \le \int_\Omega \frac{[T_k(b(x,\tilde{u}))(T)]^2}{2}dx + C\int_0^T \int_\Omega |\nabla T_k(b(x,\tilde{u}))|^p dxdt.$$

For every $z \in [0,T]$. Using (6.4.10) with $r = T$, we deduce

$$\int_\Omega \frac{[z(\tau)]^2}{2}dx + \frac{1}{2}\int_0^T \int_\Omega b_s(x,\tilde{u})|\nabla z|^2 dxdt \le Ck(\|\mu\|_{\mathcal{M}_b(Q)} + \|b(x,u_0)\|_{L^1(\Omega)} + \|B\|^p_{L^p(\Omega)}),$$

for every $z \in [0,T]$. This implies

$$\|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le CkM. \tag{6.4.14}$$

Recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$; thus

$$\|z\|_{L^p(0,T;V)} \le C(\|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|z\|^p_{L^p(0,T;L^2(\Omega))}).$$

We deduce from (6.4.14) that

$$\|z\|_{L^p(0,T;V)} \leq C[(kM)^{\frac{1}{p}} + (kM)^{\frac{1}{2}}]. \tag{6.4.15}$$

Moreover, the equation in (6.4.13) implies

$$\|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq (\|z\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))} + \|T_k(b(x,\tilde{u}))\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))}),$$

hence, using (6.4.13) and (6.4.14). We deduce

$$\|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C(kM)^{\frac{1}{p'}}. \tag{6.4.16}$$

Combining (6.4.15) and (6.4.13), we conclude that

$$\|z\|_W \leq C\max\{(kM)^{\frac{1}{p}}, (kM)^{\frac{1}{p'}}\}, \tag{6.4.17}$$

where M is defined in (6.4.12).

Step. 3 Proof completed for nonnegative data. Let us assume that $\mu \geq 0$ and $b(x,u_0) \geq 0$; hence, we have $b(x,\tilde{u})_t - \Delta_p(b(x,\tilde{u})) \geq 0$, and $b(x,\tilde{u}) \geq 0$ in Q. We claim that

$$T_k(b(x,\tilde{u}))_t - \Delta_p T_k(b(x,\tilde{u})) \geq 0. \tag{6.4.18}$$

To prove (6.4.18), we consider $S_{k,\sigma}(s)$ the smooth approximation of $T_k(s)$ and its primitive $T_{k,\sigma}(s)$. Let $\varphi \in C_c^\infty(Q)$ be a nonnegative function and take $T'_{k,\sigma}(b(x,\tilde{u}))\varphi$ as test function in (6.1.1). We obtain, using that $\mu \geq 0$ and that $T_{k,\sigma}(s)$ is concave for $s \geq 0$,

$$-\int_0^T \varphi_t T_{k,\sigma}(b(x,\tilde{u}))dt + \int_Q \tilde{a}(t,x,\nabla\tilde{u}) \cdot \nabla\varphi S_{k,\sigma}(b(x,\tilde{u}))dxdt \geq 0.$$

which yields (6.4.18) as σ goes to 0.
Combining (6.4.13) and (6.4.18), we obtain

$$-z_t - \Delta_p z \geq -(T_k(b(x,\tilde{u})))_t - \Delta_p T_k(b(x,\tilde{u})). \tag{6.4.19}$$

since both z and $T_k(b(x,\tilde{u}))$ belong to $L^p(0,T;W_0^{1,p}(\Omega))$, a standard comparison argument (multiply both sides of (6.4.19) by $(z - T_k(b(x,\tilde{u})))^-$) allows us to conclude that $z \geq T_k(b(x,\tilde{u}))$ a.e. in Q. In particular, $z \geq k$ a.e. on $\{b(x,\tilde{u}) > k\}$. On the other hand, since $\tilde{u}$ belongs to W, it has a unique cap_p quasi-continuous representative (still denoted by u), hence, the set $\{u > k\}$ is cap_p quasi-open, and its capacity can be estimated with (1.12.3). Therefore, we get

$$\mathrm{cap}_p(\{|b(x,\tilde{u})| > k\}) \leq \|\frac{z}{k}\|_W$$

using (6.4.17) we obtain the result (6.1.12).

Step. 4 Comparison with μ^+ and μ^- when μ is a smooth function. Let us consider the case where $\mu \in C^\infty(\overline{Q})$. Then $\mu^+ \in \mathcal{M}_b(Q) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$ and we can consider the unique solution $v \in W$ of the problem

$$\begin{cases} b(x,v)_t - \Delta_p v = \mu^+ & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ v = b(x,u_0)^+ & \text{on } \{0\}\times\Omega. \end{cases} \tag{6.4.20}$$

By comparison principle, we have $v \geq \tilde{u}$. Using *Step. 3* we deduce that there exists a nonnegative function $z \in W$ such that

$$z \geq T_k(b(x,v)) \geq T_k(b(x,\tilde{u}))$$

and

$$\|z\|_W \leq C\max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\},$$

where $C = C(\|\mu\|_{\mathcal{M}_b(Q)}, \|b(x,u_0)\|_{L^1(\Omega)}, p)$. Similarly, using the solutions of (6.4.20) with data $-\mu^-$ and $-b(x,u_0)^-$, we deduce that there exists a nonnegative function $w \in W$ such that

$$T_k(b(x,\tilde{u})) \geq -w$$

and

$$\|\tilde{u}\|_W \leq C\max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\}.$$

We have thus proved that there exist two nonnegative function $z, w \in W$ such that

$$-w \leq T_k(b(x,\tilde{u})) \leq z \text{ and } \|z\|_W + \|w\|_W \leq C\max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\},$$

where C depends on $\|\mu\|_{\mathcal{M}_b(Q)}$, $\|b(x,u_0)\|_{L^1(\Omega)}$ and p.

Step. 5 Proof completed. Let us fix $\Theta \in C_c^\infty(Q)$ and set $\tilde{\mu} = \Theta\mu$. By standard properties of convolution (see **[DPP]**, Lemma 2.25), given a sequence of mollifiers (ρ_n), we have $\rho_n * \tilde{\mu} \in C_c^\infty(Q)$,

$$\rho_n * \tilde{\mu} \to \tilde{\mu} \text{ strongly in } L^{p'}(0,T;W^{-1,p'}(\Omega)),$$
$$\|\rho_n * \tilde{\mu}\|_{\mathcal{M}_b(Q)} \leq \|\tilde{\mu}\|_{\mathcal{M}_b(Q)} \leq \|\mu\|_{\mathcal{M}_b(Q)}.$$

Take now $\{\Theta_j\}$ to be a sequence of $C_c^\infty(Q)$ functions such that $\Theta_j \to 1$ and consider the solutions $\tilde{u}_{j,n}$ of the problem

$$\begin{cases} b(x,\tilde{u}_{j,n})_t - \Delta_p \tilde{u}_{j,n} = \rho_n * (\Theta_j\mu) & \text{in } (0,T)\times\Omega, \\ b(x,\tilde{u}_{j,n}) = b(x,u_0) & \text{on } \{0\}\times\Omega, \\ \tilde{u}_{j,n} = 0 & \text{on } (0,T)\times\Omega. \end{cases} \tag{6.4.21}$$

As $n \to \infty$, the sequence $(\tilde{u}_{j,n})$ converges in $L^p(0,T;W_0^{1,p}(\Omega))$ to the solution $\tilde{u}_j$ of (6.1.1) with $\Theta_j\mu$ as datum. Next, as $j \to +\infty$,

$$\tilde{u}_j \to \tilde{u} \text{ in } L^\infty(0,T;L^1(\Omega)).$$

This is consequence of a standard L^1-contraction argument. Indeed, subtracting equations (6.1.1) and (6.4.21), and taking $T_k(\tilde{u}_{j,n} - \tilde{u})$ as test function, we get (note that both $\tilde{u}_{j,n}$ and $\tilde{u}$ belong to W)

$$\int_\Omega |\tilde{u}_{j,n} - \tilde{u}|(t)dx \leq C\|\rho_n * (\Theta_j\mu - \Theta_j\mu)\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\|T_1(\tilde{u}_{j,n} - \tilde{u})\|_{L^p(0,T;W_0^{1,p}(\Omega))}$$
$$+ C\int_\Omega T_1(\tilde{u}_{j,n} - \tilde{u})(\Theta_j - 1)d\mu$$

which yields

$$\|(\tilde{u}_{j,n} - \tilde{u})(t)\|_{L^1(\Omega)} \leq C\|\rho_n * (\Theta_j\mu) - \Theta_j\mu\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\|T_1(\tilde{u}_{j,n} - \tilde{u})\|_{L^p(0,T;W_0^{1,p}(\Omega))}$$
$$+ C\|(1-\Theta_j)\mu\|_{\mathcal{M}_b(Q)}.$$

Since for j fixed $\tilde{u}_{j,n}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ as $n \to +\infty$, the first term with right-hand side tends to 0, hence

$$\|(\tilde{u}_j - \tilde{u}\|(t)\|_{L^1(\Omega)} \leq C\|(1-\Theta_j)\mu\|_{\mathcal{M}_b(Q)}.$$

Since the later term tends to zero as $j \to \infty$ by dominated convergence, we deduce the convergence of $\tilde{u}_j$ to $\tilde{u}$. By *Step. 4*, there exist a nonnegative function $z_{j,n}$ and $w_{j,n}$ such that

$$-w_{j,n} \leq T_k(b(x,\tilde{u}_{j,n})) \leq z_{j,n}$$

and

$$\|z_{j,n}\|_W + \|w_{j,n}\|_W \leq C\max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\},$$

where $C = C(\|\rho_n * (\Theta_j\mu))\|_{\mathcal{M}_b(Q)}, \|b(x,u_0)\|_{L^1(\Omega)}, p)$. Since

$$\|\rho_n * (\Theta_j\mu)\|_{\mathcal{M}_b(Q)} \leq \|\mu\|_{\mathcal{M}_b(Q)},$$

the constant C can be chosen independently of n and j. The sequences $(z_{j,n})$ and $(w_{j,n})$ being bounded in W, they converge weakly up to subsequences to nonnegative functions $z, w \in W$ and almost everywhere in Q. Thus,

$$-w \leq T_k(b(x,\tilde{u})) \leq z \text{ a.e. in } Q$$

and

$$\|z\|_W + \|w\|_W \leq C\max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\}.$$

where $C = C(\|\mu\|_{\mathcal{M}_b(Q)}, \|b(x,u_0)\|_{L^1(\Omega)}, p)$. Since $\tilde{u} \in W$, it admits a uniquely defined cap_p quasi-continuous representative; hence, the sets $\{\tilde{u} > k\}$ and $\{\tilde{u} < -k\}$ are cap_p quasi-open. Using (1.12.3), we get

$$\text{cap}_p(\{|b(x,\tilde{u})| > k)\}) \leq \text{cap}_p(\{b(x,\tilde{u}) > k\}) + \text{cap}_p(\{|\tilde{u}| < -k\}) \leq \|\frac{z}{k}\|_W + \|\frac{w}{k}\|_W$$

which yields the result (6.1.12) for $u = \tilde{u}$.

CHAPTER 7

Nonlinear parabolic problems with absorption term and singular measure data

The problem of the nonexistence, the so-called *absorption* problem, has been the subject of several works. We can cite in the elliptic framework the results of [**Br1, BB, BBC**], of [**BGO2, CN, OP**] in the nonlinear framework, [**Pe2, BidV1**] in the case of parabolic problems and [**BidVP**] for systems. In the case of removable singularities, the number of publications is so great that we cannot cite all of them; let us only mention [**BV, BidV, ML, BrN**] in the case of equations and [**SZ**] for systems. Recall that such results can be used for finding a priori estimates and nonexistence results in bounded domains via a blow-up technique. Obtaining a priori estimates for general spaces is most often difficult, even in the case of Orlicz-Sobolev spaces (see Section 1.18) and many questions are still open. The main results can be found in [**FG, F, FP**], and also [**Ais, Ais1, Ais3, AisB1**]. Let us give an example showing the connections between equations and inequalities in order to discuss the application of the notion of capacity related to a nonexistence result of solutions for some nonlinear parabolic equations having absorption term and measure data. Assume that $N \geq 3$ and $q > 1$, we study the nonexistence of solutions for the following nonlinear parabolic equation whose model is

$$\begin{cases} u_t - \operatorname{div}(a(t,x,\nabla u)) + |u|^{q-1}u = g + \lambda & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.0.1}$$

where $1 < p < N$, $g \in L^1(Q)$, λ is a measure concentrated on a set of zero parabolic $r-$capacity and $u \mapsto -\operatorname{div}(a(t,x,\nabla u))$ is a pseudo-monotone operator and consider the corresponding bilateral obstacle problem with measure data concentrated on a set of zero parabolic $p-$capacity whose model is

$$\begin{cases} \langle u_t - \operatorname{div}(a(t,x,\nabla u)) - \lambda,\ v - u\rangle \geq 0, \\ u \in K = \{w \in L^p(0,T;W_0^{1,p}(\Omega)) : |w| \leq 1\} \text{ for every } v \in K, \end{cases} \tag{7.0.2}$$

using a notion of entropy solutions with convergence properties essential to establish a non-stability result. This leads us to come back to the problem where Ω be a bounded open subset of $\mathbb{R}^N$, $N > 2$, with $0 \in \Omega$, f a function in $L^1(\Omega)$, and f_n be a sequence of $L^\infty(\Omega)-$functions such that $\lim\limits_{n\to+\infty} \int_{\Omega\setminus B_\rho(0)} |f_n - f|dx = 0$ for all $\rho > 0$ with u_n be the sequence of solutions of the nonlinear elliptic problems $-\Delta u_n + |u_n|^{q-1}u_n = f_n$ in Ω with $q \geq \frac{N}{N-1}$; then u_n converges to the unique solution u of the equation $-\Delta u + |u|^{q-1}u = f$ (see Section 7.1). The result of [**Br1**] is strongly connected with a theorem by Bénilan and Brezis [**BB**], which states that the problem $-\Delta u + |u|^{q-1}u = \delta_0$ has a distributional solution if $q \geq \frac{N}{N-2}$. On the other hand, if $q < \frac{N}{N-2}$ [**Br1, BBC**]; then there exists a unique solution of $-\Delta u + |u|^{q-1}u = \delta_0$ in Ω. Thus the preceding result can be seen as a nonexistence theorem. The dividing range "$\frac{N}{N-2}$" basically depends on two facts: the linearity of the Laplacian operator (i.e., the dependence of order 1 with respect to the gradient of u), and the fact that the Dirac δ_0 is a measure which in concentrated on a point (a set of zero $N-$capacity). In the case $q \geq \frac{N}{N-2}$, which is equivalent to $2q' \leq N$, δ_0 is not "absolutely continuous" with respect to the $N-$capacity and hence also to the $2q'-$capacity and there is no solution. If $q < \frac{N}{N-2}$, which is equivalent to $2q' > N$, δ_0 is "absolutely continuous" with respect to the $2q'-$capacity and there is a solution. In [**OP**], this result was improved to the nonlinear framework, where the authors actually proved that, if λ is a measure concentrated on a set of zero $r-$capacity, $r < q$, and q large enough, then problem $-\Delta_p u + |u|^{q-1}u = \lambda$ in Ω has no solutions in a very strong sense, that is, if we approximate λ with smooth function in the narrow topology of measures then approximate solution

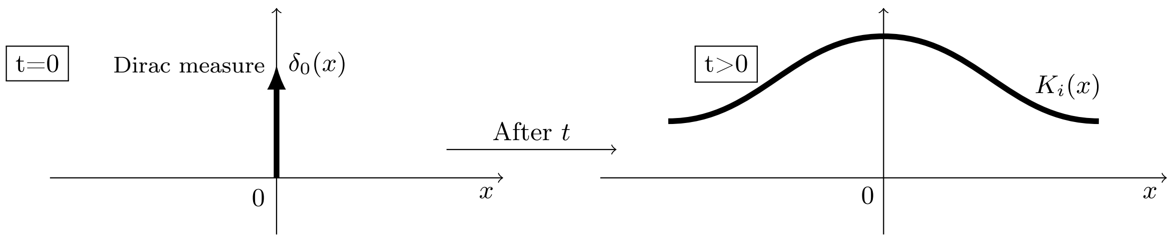


FIGURE 22. The heat Kernel of Dirac mass δ_0

u_n converge to 0. In the same paper the result is proved for more general Leray-Lions type operators. The result of [**OP**] has been extended to nonlinear parabolic operators with measures concentrated on sets of null $r-$capacity in [**Pe2**]. The plan of this Chapter is as follows: In Section 7.1, we details some known results about nonexistence theorems. Section 7.2 contains some notations on the $r-$capacity and the main assumptions. In Section 7.3, we briefly sketch the proof of nonexistence result of problem (7.0.1) and we prove the same result for the corresponding bilateral obstacle problem (7.0.2) in Section 7.4.

7.1. Classification of some preliminary results

As we said before, we study the non-stability of solutions and the question of removable sets $E \subset Q$ in terms of capacity conditions on λ and E. This leads us to come back to the problem without perturbation and measure terms, i.e.,

$$u_t - \operatorname{div}(a(t,x,\nabla u)) = g, \quad \text{in } Q$$

for which we define a notion of entropy solution, and we give convergence properties, essential to our proofs. Recall some preliminary results on similar elliptic and parabolic problems. Note that the first question is to find conditions on q and r which ensure the nonexistence of solutions. In the case $p = 2$, a necessary and sufficient condition was found in [**BB**] for the problem with absorption and δ_0 as data (Dirac measure concentrated at sets of zero $N-$capacity)

$$\begin{cases} -\Delta u + |u|^{q-1}u = \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

this problems has no distributional solution if $q \geq \frac{N}{N-2}$. On the other hand, there exist a (weak) solution if and only if $q < \frac{N}{N-2}$ [**Br1, BBC**]. We have the following Theorem proved in [**Br1**]

THEOREM 7.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N > 2$, with $0 \in \Omega$, let $f \in L^1(\Omega)$ and f_n be a sequence of $L^\infty(\Omega)-$functions such that*

$$\lim_{n\to\infty} \int_{\Omega \setminus B_\rho(0)} |f_n - f| dx = 0, \quad \forall \rho > 0. \tag{7.1.1}$$

Let u_n be solutions of the nonlinear elliptic problems (with $q \geq \frac{N}{N-2}$)

$$\begin{cases} -\Delta u_n + |u_n|^{q-1}u_n = f_n & \textit{in } \Omega, \\ u_n = 0 & \textit{on } \partial\Omega. \end{cases} \tag{7.1.2}$$

Then, u_n converges to the unique solution u of the equation $-\Delta u + |u|^{q-1}u = f$.

If $f = 0$, the sequence of $L^\infty(\Omega)-$functions converging in the weak–* topology of measures to δ_0 are an example of f_n. In the case of problems with measures λ concentrated on sets of zero $r-$capacity, a conditions on q and r are also necessary. A precise and sufficient conditions was given in [**BPi, GM**] for problems

$$\begin{cases} -\Delta u + |u|^{q-1}u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ belongs to $L^1(\Omega) + W^{-2,q}(\Omega)$, it is equivalent to say that μ is "absolutely continuous" with respect to the $(2, q')-$capacity such that

$$\text{cap}_{2q'+\epsilon}(E) = 0 \Longrightarrow \text{cap}_{2,q'}(E) = 0 \quad \forall \epsilon > 0,$$

see **[AH]**. It implies in particular that, if μ is concentrated on a set of $r-$capacity zero, and $r > 2q'$ (i.e., $q > \frac{r}{r-2}$), then μ is not absolutely continuous with respect to the $(2, q')-$capacity. In the case $p \neq 2$, the question becomes more difficult, because the non-linearity of the divergentiel operator (i.e., the dependence of order $p-1$ with respect to the gradient of u), and the fact that the measure is singular, that means, a special type of suitable test functions "cut-off functions" to deal with measures (see **[BGO2, DMOP, Po, VV]**). Concerning problem (7.0.1) with Dirichlet boundary, it was recently shown in **[Pe2]** that is: if λ is concentrated on sets of zero parabolic $r-$capacity, for some $r > p > 1$, and q large enough, then sequences of approximate solutions do not converge to a "reasonable" solution. This suggested that in some sense problem (7.0.1) might have no solution. Using the notion of entropy solution, the result is might true, for singular measures, and much more general.

THEOREM 7.2. *Let $1 < p < r$, $q > \frac{r(p-1)}{r-p}$, and let u_n the unique solution of problem*

$$\begin{cases} (u_n)_t - \text{div}(a(t, x, \nabla u_n)) + |u_n|^{q-1}u_n = g_n + f_n & \textit{in } (0, T) \times \Omega, \\ u_n(t, x) = 0 & \textit{on } (0, T) \times \partial\Omega, \\ u_n(0, x) = 0 & \textit{in } \Omega. \end{cases} \tag{7.1.3}$$

Then, $|\nabla u_n|^{p-1}$ converges strongly to $|\nabla u|^{p-1}$ in $L^\sigma(Q)$ with $\sigma < \frac{pq}{(q+1)(p-1)}$, where u is the unique entropy (renormalized) solution of problem

$$\begin{cases} u_t - \text{div}(a(t, x, \nabla u)) + |u|^{q-1}u = g & \textit{in } (0, T) \times \Omega, \\ u(t, x) = 0 & \textit{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0 & \textit{in } \Omega. \end{cases} \tag{7.1.4}$$

Moreover

$$\lim_{n\to\infty} \int_Q |u_n|^{q-1}u_n\varphi dx = \int_Q |u|^{q-1}u\varphi dx + \int_Q \varphi d\lambda, \quad \forall \varphi \in C_0(Q). \tag{7.1.5}$$

Notice that we have no restriction of the sign of u and λ and this result concerns the case $q > \frac{r(p-1)}{r-p}$, where $r > p > 1$. In the case $q = \frac{N(p-1)}{N-1}$, this is a result of removable singularities. In particular, problems (7.0.1) have no solution if λ concentrated on points. recall that when $p = 2$, we have a stronger result for the problem (7.0.1) with source term λ, which has to compared to the one of **[Br1]**.

THEOREM 7.3. *Let f_n be a sequence of functions in $L^\infty(Q)$ such that*

$$\lim_{n\to\infty} \int_Q \varphi f_n dx = \int_Q \varphi d\lambda \quad \forall \varphi \in C(\bar{Q}),$$

where λ is a bounded Radon measure on Q concentrated on a set of zero parabolic $r-$capacity, and let $q > \frac{r}{r-2}$. Then the solutions of

$$\begin{cases} (u_n)_t - \Delta u_n + |u_n|^{q-1}u_n = f_n & \textit{in } (0, T) \times \Omega, \\ u_n(t, x) = 0 & \textit{on } (0, T) \times \partial\Omega, \\ u_n(0, x) = 0 & \textit{in } \Omega, \end{cases}$$

are such that, both u_n and $|\nabla u_n|$ converges to 0 in $L^1(Q)$. Moreover,

$$\lim_{n\to\infty} \int_Q |u_n|^{q-1}u_n\varphi dx = \int_Q \varphi d\lambda, \quad \forall \varphi \in C_0(Q).$$

Now we come back to our question, namely the characterization of the sets of Radon measures such that the variational inequality

$$\langle u_t - \text{div}(a(t, x, \nabla u)) - \lambda, v - u\rangle \geq 0 \tag{7.1.6}$$

has a solution. For linear elliptic operators ($p = 2$) it was shown in **[DLeo]** by means of duality arguments and by a new definition of solution, if the measure λ is concentrated on a set of zero $2-$capacity, the solution founded is zero. If $p \neq 2$, the decomposition result of **[FST]** suggest that measures concentrated on sets of zero $p-$capacity "disappear" passing to the limit in the approximation process. We recall that it is also true for $L^1(\Omega)$ data **[DO3]**.

THEOREM 7.4. *Let g be a function in $L^1(\Omega)$, and let (u_n) be the sequence of entropy solutions of the following problem*

$$\begin{cases} Au_m + |u_m|^{m-1}u_m = g & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega. \end{cases}$$

Then u_m converges to u as m tends to infinity, where u is the unique solution of the variational inequality

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla u) \cdot \nabla(v - u)dx \geq \int_\Omega g(v - u)dx \quad \forall v \in K, \\ u \in K = \{w \in W_0^{1,p}(\Omega) : |w| \leq 1\}. \end{cases}$$

Thus, in particular for $g \in W^{-1,p'}(\Omega)$ (see **[BM1, DO3]**). It applies also to problems with measure data which is concentrated on a set of zero $p-$capacity plus a function in $L^1(\Omega)$ **[DO3]**.

THEOREM 7.5. *Let g be a function in $L^1(\Omega)$, G be an element of $(L^{p'}(\Omega))^N$, $\lambda = \lambda^+ - \lambda^-$ be a bounded Radon measure concentrated on a set E of zero $p-$capacity and $f_n = f_n^{\oplus} - f_n^{\ominus}$ be a sequence of $L^\infty(\Omega)$ functions that converges to λ. Let u_n be the unique solution of the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla u_n) \cdot \nabla(v - u_n)dx \geq \int_\Omega g(v - u_n)dx + \int_\Omega G \cdot \nabla(v - u)dx + \int_\Omega f_n(v - u_n) \quad \forall v \in K, \\ u \in K = \{w \in W_0^{1,p}(\Omega) : |w| \leq 1\}. \end{cases}$$

Then u_n converges strongly in $W_0^{1,p}(\Omega)$ to u as n tends to infinity, where u is the unique solution of the variational inequality

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla u) \cdot \nabla(v - u)dx \geq \int_\Omega g(v - u)dx + \int_\Omega G \cdot \nabla(v - u)dx \quad \forall v \in K, \\ u \in K = \{w \in W_0^{1,p}(\Omega) : |w| \leq 1\}. \end{cases}$$

Note that these results are based on a priori estimates of the solution given in **[B6, DMOP]**. Now consider a right hand side of the form $\mu = g_1 - \operatorname{div}(G) + g_2^t + \lambda$ and discuss the question of inequalities corresponding to the problem of the type

$$\begin{cases} u_t - \operatorname{div}(a(t, x, \nabla u)) + |u|^{q-1}u = \mu & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

REMARK 7.6. If the convergence is stronger than the one stated in Theorem 7.3 and the Laplacian operator substituted by a more general nonlinear monotone operators of the order $p - 1$, $|\nabla u_n|^{p-1}$ converges to $|\nabla u|^{p-1}$ in $L^\sigma(Q)$ with $\sigma < \frac{pq}{(q-1)(p-1)}$, where u is the unique solution of the problem

$$\begin{cases} u_t - \operatorname{div}(a(t, x, \nabla u)) + |u|^{q-1}u = g & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0) = 0 & \text{in } \Omega. \end{cases}$$

under the conditions $1 < p < r$ and $q > \frac{r(p-1)}{r-p}$.

And the question now is the following: suppose that we have a measure λ which is concentrated on a set E of zero $p-$capacity and a function g in $L^1(Q)$; suppose we have a sequence $\{f_n\}$ of functions which converges to λ in the weak–* topology of measures, and a sequence g_n which converges to g in $L^1(Q)$. The result of Theorem 7.2 holds true for the corresponding variational inequality? In the next, we will give an answer to the question

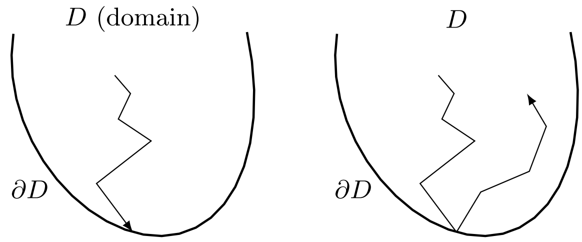


FIGURE 23. The absorption (reflection) phenomenon

using the particular sequence of cut-off functions and we deal with a general case of problem (7.0.1) with a zero lower order term. We shall prove the following result

THEOREM 7.7. *Let $g_1 \in L^1(Q)$, G be an element of $(L^{p'}(Q))^N$ and $g_2^t \in L^p(0,T;V)$. Let $\lambda = \lambda^+ - \lambda^-$ be a bounded Radon measure concentrated on a set E of zero $p-$capacity. Let $f_n = f_n^{\oplus} - f_n^{\ominus}$ be a sequence of $L^\infty(Q)-$functions which converges to $\pm\lambda$ in the sense of*

$$\lim_{n\to+\infty} \int_Q f_n^{\oplus}\varphi dx = \int_Q \varphi d\lambda^+, \quad \lim_{n\to+\infty} \int_Q f_n^{\ominus}\varphi dx = \int_Q \varphi d\lambda^- \tag{7.1.7}$$

for every function φ which is continuous and bounded on Q. Let u_n be the unique solution of the variational inequality

$$\begin{cases} \displaystyle\int_0^T \langle (u_n)_t, v-u_n\rangle dt + \int_Q a(t,x,\nabla u)\cdot\nabla(v-u_n)dxdt \\ \displaystyle\geq \int_Q g_1(v-u_n)dxdt - \int_Q G\cdot\nabla(v-u_n)dxdt + \int_0^T \langle g_2^t, v-u_n\rangle dt + \int_Q f_n(v-u_n)dxdt \quad \forall v\in K \\ u_n \in K = \{w \in L^p(0,T;W_0^{1,p}(\Omega)) : |w|\leq 1 \ a.e. \ in \ Q\} \end{cases} \tag{7.1.8}$$

Then u_n converges strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ to u as n tends to infinity, where u is the unique solution of the variational inequality

$$\begin{cases} \displaystyle\int_0^T \langle u_t, v-u\rangle dt + \int_Q a(t,x,\nabla u)\cdot\nabla(v-u)dxdt \\ \displaystyle\geq \int_Q g_1(v-u)dxdt - \int_Q G\cdot\nabla(v-u)dxdt + \int_0^T \langle g_2^t, v-u\rangle dt \quad \forall v\in K \\ u_n \in K = \{w \in L^p(0,T;W_0^{1,p}(\Omega)) : |w|\leq 1 \ a.e. \ in \ Q\} \end{cases} \tag{7.1.9}$$

REMARK 7.8. We explicitly remark that $f_n^{\oplus}$ and $f_n^{\ominus}$ may not be the positive and negative parts of f_n (that is to say, their supports may not be disjoint). Observe that choosing $\varphi \equiv 1$ in (7.1.7) we obtain

$$\|f_n^{\oplus}\|_{L^1(Q)} \leq C, \quad \|f_n^{\ominus}\|_{L^1(Q)} \leq C. \tag{7.1.10}$$

As a consequence of the previous theorem, the measures concentrated on sets of zero $p-$capacity "disappear" passing to the limit in the approximation process. This fact will allow us to characterize the measures for which the variational inequality has a "standard" solution. In the following, we define $\omega(n,m,\delta,\eta)$ any quantity (depending on n,m,δ and η) such that $\lim\limits_{\delta\to 0^+}\lim\limits_{\eta\to 0^+}\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}|\omega(n,m,\delta,\eta)| = 0$. Similarly, if the quantity we are considering does not depend one or more of the three four parameters n,m,δ and η, we will omit the dependence from it in ω, for example, $\omega(m,\delta,\eta)$ is any quantity such that $\lim\limits_{\delta\to 0^+}\lim\limits_{\eta\to 0^+}\lim\limits_{m\to\infty}|\omega(m,\delta,\eta)| = 0$. Finally C will be a constant that may change from an inequality to another to indicate a dependence of C on the real parameters δ we shall write $C = C(\delta)$.

7.2. Main sssumptions and entropy formulation

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$, $T > 0$ and Q be the cylinder $(0,T) \times \Omega$. We are interested in the non-stability results of solutions for bilateral obstacle problems with measures as data corresponding to the general variational equality

$$\begin{cases} u_t - \operatorname{div}(a(t,x,\nabla u)) + |u|^{q-1}u = g + \lambda, & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.2.1}$$

where $q > 1$, $1 < p < N$, $g \in L^1(Q)$ and λ is a measure concentrated on a set of zero $r-$capacity. The function $a : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, $a(\cdot,\cdot,\zeta)$ is measurable on Q for every ζ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q) satisfying the following assumptions:

$$a(t,x,\zeta) \cdot \zeta \geq c_0|\zeta|^p, \quad c_0 > 0, \tag{7.2.2}$$

$$|a(t,x,\zeta)| \leq b_0(t,x) + c_1|\zeta|^{p-1}, \quad c_1 > 0, \tag{7.2.3}$$

$$\langle a(t,x,\zeta) - a(t,x,\zeta') \cdot (\zeta - \zeta')\rangle > 0, \quad \zeta \neq \zeta', \tag{7.2.4}$$

for almost every $(t,x) \in Q$ and for every $\zeta, \zeta' \in \mathbb{R}^N$, $b_0(t,x)$ is a nonnegative function in $L^{p'}(Q)$. The map $u \mapsto -\operatorname{div}(a(t,x,\nabla u))$ is a coercive, continuous, bounded and monotone operator defined from $L^p(0,T;W_0^{1,p}(\Omega))$ with values in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We first recall some notations and definitions, let $p > 1$, we recall the notion of parabolic $r-$capacity associated to W_r (see definition of W with p replaced by r), for any $r > 1$, is defined by

$$\operatorname{cap}_r(K,Q) = \inf\{\|u\|_{W_r} : u \in C_c^\infty(Q);\ u \geq \chi_K\}, \tag{7.2.5}$$

for any compact set $K \subset Q$. In the sequel we set $q > \frac{r(p-1)}{r-p}$, so that $q > \frac{r}{r-2}$ when $p = 2$. Let us recall that a sequence (λ_n) of measures in $\mathcal{M}_b(Q)$ converges in the narrow topology to a measure λ in $\mathcal{M}_b(Q)$ if

$$\lim_{n\to+\infty} \int_Q \varphi d\lambda_n = \int_Q \varphi d\lambda \tag{7.2.6}$$

for every $\varphi \in C_b(Q)$. In order to localize some integral near the support of the singular measure μ_s with respect to $p-$capacity. Let us consider the space

$$S = \{u \in L^p(0,T;W_0^{1,p}(\Omega));\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\},$$

endowed with its natural norm $\|u\|_S = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$, and its subspace W_2 as

$$W_2 = \{u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q);\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\},$$

endowed with its natural norm

$$\|u\|_{W_2} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u\|_{L^\infty(Q)} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}.$$

Most part of this Chapter will be concerned with the proof of Theorem 7.7. The notion of entropy solution for the parabolic problem will be given as a natural extension of the one of the elliptic case (see for instance **[B6, Pr2]**).

Definition 7.9. Let $\mu_0 \in \mathcal{M}_0(Q)$ and $\lambda = 0$. A measurable function u is an entropy solution of

$$\begin{cases} u_t - \operatorname{div}(a(t,x,\nabla u)) + |u|^{q-1}u = \mu_0 + \lambda & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.2.7}$$

if

(a1) $T_k(u-g) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$,

(b1) $t \in [0,T] \mapsto \int_\Omega \Theta_k(u-g-\varphi)(t,x)dx$ is continuous function for all $k \geq 0$ and all $\varphi \in S^p \cap L^\infty(Q)$, (c1) for all $k \geq 0$ and all $\varphi \in S^p \cap L^\infty(Q)$

$$\int_\Omega \Theta_k(u - g_2 - \varphi)(T,x)dx - \int_\Omega \Theta_k(u - g_2 - \varphi)(0,x)dx + \int_0^T \langle \varphi_t, T_k(u - g_2 - \varphi)\rangle dt$$
$$+ \int_Q a(t,x,\nabla u) \cdot \nabla T_k(u - g_2 - \varphi)dxdt \leq \int_Q g_1 T_k(u - g - \varphi)dxdt + \int_\Omega G_1 \cdot \nabla(T_k(u - g_2 - \varphi))dxdt.$$

We approximate the data with smooth μ_0^n which converge to μ_0 in $\mathcal{M}_b(Q)$ and smooth $f_n = f_n^\oplus - f_n^\ominus$, with $f_n^\oplus$ and $f_n^\ominus$ converging respectively, to λ^+ and λ^- in the narrow topology of measures. We consider the solutions u_n of

$$\begin{cases} (u_n)_t - \mathrm{div}(a(t,x,\nabla u_n)) + |u_n|^{q-1}u_n = \mu_0^n + f_n & \text{in } Q := \Omega \times (0,T), \\ u_n = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_n(0) = u_0 & \text{in } \Omega. \end{cases} \tag{7.2.8}$$

Thanks to Definition 7.9, it is possible to prove the equivalence of the unique entropy solution of problem (7.2.7) (with $\lambda = 0$) with the renormalized solution of the same problem. Moreover notice that the argument in [**DP**] allow us to deduce that a entropy solution turns out to coincide with renormalized solution even for diffuse measures.

LEMMA 7.10. *Let $\mu = \lambda_s^+ - \lambda_s^-$ be a bounded radon measure on Q, where λ_s^+ and λ_s^- are non-negative and concentrated, respectively, on two disjoint sets E^+ and E^- of zero $r-$capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\lambda_s^+(E^+ \backslash K_\delta^+) \leq \delta, \quad \lambda_s^-(E^- \backslash K_\delta^-) \leq \delta \tag{7.2.9}$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_0^1(Q)$, such that

$$\begin{aligned} &\psi_\delta^+, \psi_\delta^- \equiv 1 \textit{ respectively on } K_\delta^+, K_\delta^-, \\ &0 \leq \psi_\delta^+, \psi_\delta^- \leq 1, \\ &\mathit{Supp}(\psi_\delta^+) \cap \mathit{Supp}(\psi_\delta^-) \equiv \emptyset. \end{aligned} \tag{7.2.10}$$

Moreover

$$\|\psi_\delta^+\|_{S^r} \leq \delta, \quad \|\psi_\delta^-\|_{S^r} \leq \delta,$$

and in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and decomposition of $(\psi_\delta^-)_t$ such that

$$\begin{aligned} &\|(\psi_\delta^+)_t\|_{L^{r'}(0,T;W^{-1,r'}(\Omega))} \leq \delta, \quad \|(\psi_\delta^+)_t\|_{L^1(Q)} \leq \delta, \\ &\|(\psi_\delta^-)_t\|_{L^{r'}(0,T;W^{-1,r'}(\Omega))} \leq \delta, \quad \|(\psi_\delta^-)_t\|_{L^1(Q)} \leq \delta, \end{aligned} \tag{7.2.11}$$

and both ψ_δ^+ and ψ_δ^- converge to zero weakly$-*$ in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes. Moreover, if $f_n = f_n^\oplus - f_n^\ominus$ is as in (7.1.7), we have

$$\begin{aligned} &\int_Q \psi_\delta^- f_n^\oplus = \omega(n,\delta), \quad \int_Q \psi_\delta^- d\lambda_s^+ \leq \delta, \\ &\int_Q \psi_\delta^+ f_n^\ominus = \omega(n,\delta), \quad \int_Q \psi_\delta^+ d\lambda_s^- \leq \delta, \\ &\int_Q (1 - \psi_\delta^+) f_n^\oplus = \omega(n,\delta), \quad \int_Q (1 - \psi_\delta^+) d\lambda_s^+ \leq \delta, \\ &\int_Q (1 - \psi_\delta^-) f_n^\ominus = \omega(n,\delta), \quad \int_Q (1 - \psi_\delta^-) d\lambda_s^- \leq \delta, \end{aligned} \tag{7.2.12}$$

PROOF. We follow the lines of [**DMOP, Pe1**]. We recall that λ^+ and λ^- are concentrated on two disjoint subsets E^+ and E^- whose $r-$capacity is zero. Moreover, since λ^+ and λ^- are Radon measures, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that

$$\lambda^+(E^+ \backslash K_\delta^+) \leq \delta, \qquad \lambda^-(E^- \backslash K_\delta^-) \leq \delta.$$

Since $K_\delta^+ \cap K_\delta^- = \emptyset$, there exist two disjoint open subsets A_δ^+ and A_δ^- such that $K_\delta^+ \subseteq A_\delta^+$ (resp. $K_\delta^- \subseteq A_\delta^-$). Moreover, since λ^+ and λ^- are Radon measures, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that

$$\lambda^+(E^+ \backslash K_\delta^+) \le \delta, \quad \lambda^-(E^- \backslash K_\delta^-) \le \delta$$

since $K_\delta^+ \cap K_\delta^- = \emptyset$, there exist two open subsets A_δ^+ and A_δ^-, disjoint, containing respectively, K_δ^+ and K_δ^- such that $K_\delta^\pm \subseteq A_\delta^\pm$. Moreover, since $\mathrm{cap}_r(K_\delta^+, Q) = 0$) (resp. $\mathrm{cap}_r(K_\delta^-, Q) = 0$), we have that $\mathrm{cap}_r(K_\delta^+, U_\delta^+) = 0$ (resp. $\mathrm{cap}_r(K_\delta^-, U_\delta^-) = 0$) (see **[Pe1]**, Lemma 4). Thus, by definition of parabolic $r-$capacity, there exist two functions $\varphi_\delta^+ \in C_0^\infty(U_\delta^+)$ (resp. $\varphi_\delta^- \in C_0^\infty(U_\delta^-)$) such that for every $\delta' > 0$,

$$\|\varphi_\delta^+\|_W \le \delta' \text{ and } \varphi_\delta^+ \ge \chi_{K_\delta^+} \text{ resp. } \|\varphi_\delta^-\|_W \le \delta' \text{ and } \varphi_\delta^- \ge \chi_{K_\delta^-}).$$

Then we obtain (7.2.9) by taking $\psi_\delta^+ = \overline{H}(\varphi_\delta^+)$ (resp. $\psi_\delta^- = \overline{H}(\varphi_\delta^-)$) with ($H(s) = 4/3$ if $|s| \le 1/2$, 0 if $|s| > 1$, and affine if $1/2 < |s| \le 1$). Moreover, we have

$$\begin{aligned} 0 \le \int_Q \psi_\delta^- \, d\lambda^+ &= \int_{A_\delta^-} \psi_\delta^- \, d\lambda^+ \le \lambda^+(A_\delta^-) \le \lambda^+(Q \backslash A_\delta^+) \\ &\le \lambda^+(Q \backslash K_\delta^+) = \lambda^+(E^+ \backslash K_\delta^+) \le \delta \end{aligned}$$

analogously

$$\int_Q \psi_\delta^+ \, d\lambda^- \le \delta.$$

Now let $\delta, \eta > 0$ fixed, we have

$$\begin{aligned} 0 \le \int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\lambda^+ &\le \int_{Q \backslash (K_\delta^+ \cap K_\eta^+)} (1 - \psi_\delta^+) d\lambda^+ \le \lambda^+(Q \backslash (K_\delta^+ \cap K_\eta^+)) \\ &\le \lambda^+(Q \backslash K_\delta^+) + \lambda^+(Q \backslash K_\eta^+) \le \delta + \eta. \end{aligned}$$

A similar result is obtained for the second inequality (7.2.11). □

REMARK 7.11. If E^+ or (E^-) is closed (hence compact), we can choose $K_\delta^+ = E^+$ ($K_\delta^- = E^-$) for $\delta > 0$. If for example $\lambda^+ = 0$, then we choose $K_\delta^+ = \emptyset$, and $\psi_\delta^+ \equiv 0$.

REMARK 7.12. Observe that as a consequence of Lemma 7.10, we have that both ψ_δ^+ and ψ_δ^- converge to zero as δ tends to zero, strongly in S^r, weakly–$*$ in $L^\infty(Q)$ and almost everywhere in Q.

Let us recall the definition of Marcinkiewicz spaces, also called weak Lebesgue spaces.

DEFINITION 7.13. Let ρ be a positive number. The Marcinkiewicz space $\mathcal{M}^\rho(\Omega)$ is the set of all measurable functions u on Ω such that

$$\text{meas}\{|u| \ge k\} \le \frac{C}{k^\rho} \text{ for every } k > 0, \tag{7.2.13}$$

for some positive constant $C > 0$. We recall that if $\text{meas}(\Omega) < +\infty$, then

$$L^\rho(Q) \subset M^\rho(Q) \subset L^{\rho-\epsilon}(Q). \tag{7.2.14}$$

with continuous embedding, for every $\rho > 1$ and for every ϵ in $(0, \rho - 1)$.

The following two results are rather technical and will be used in the proof of Theorems 7.2 and 7.7.

LEMMA 7.14. *Let $\rho > 0$ and let $\{v_n\}$ be a sequence of functions bounded in $\mathcal{M}^\rho(Q)$. Suppose that, for every $k > 0$, we have*

$$\int_Q |\nabla T_k(v_n)|^p dxdt \le Ck,$$

for some positive constant C. Then $\{|\nabla v_n|\}$ is bounded in $\mathcal{M}^s(Q)$, with $s = \frac{p\rho}{\rho+1}$.

Proof. We follow the lines of the proof of [**DMOP**], Lemma 4.2. Let σ be a fixed positive real number, we have, for every $k>0$

$$\begin{aligned}\text{meas}\{|\nabla v_n|>\sigma\} &= \text{meas}\{|\nabla v_n|>\sigma;\ |v_n|\le k\}+\text{meas}\{|\nabla v_n|>\sigma;\ |v_n|>k\}\\ &\le \text{meas}\{|\nabla v_n|>\sigma;\ |v_n|\le k\}+\text{meas}\{|v_n|>k\}.\end{aligned} \tag{7.2.15}$$

Moreover

$$\begin{aligned}\text{meas}\{|\nabla v_n|>\sigma,|v_n|\le k\} &\le \frac{1}{\sigma^p}\int_{\{|\nabla v_n|>\sigma;\ |v_n|\le k\}}|\nabla v_n|^p dx\\ &= \frac{1}{\sigma^p}\int_{\{|v_n|\le k\}}|\nabla v_n|^p dx=\frac{1}{\sigma^p}\int_Q|\nabla T_k(v_n)|^p dx\le \frac{Ck}{\sigma^p}\end{aligned}$$

Since by assumptions on v_n there exists a positive constant $\hat{C}$ such that

$$\text{meas}\{|\nabla v_n|>\sigma;\ |v_n|>k\}\le \text{meas}\{|v_n|\ge k\}\le \frac{\hat{C}}{k^\rho} \tag{7.2.16}$$

equation (7.2.16) then implies

$$\text{meas}\{|\nabla v_n|>\sigma\}\le \frac{Ck}{\sigma^\rho}+\frac{\hat{C}}{k^\rho}, \tag{7.2.17}$$

and this latter inequality holds for every $k>0$. Minimizing on k, the minimum is achieved for the value $k_0=(\frac{\rho C}{\hat{C}})^{\frac{1}{\rho+1}}\sigma^{\frac{p}{\rho+1}}$, we easily get

$$\text{meas}\{|\nabla v_n|>\sigma\}\le \frac{C}{\sigma^{\frac{p\rho}{\rho+1}}}, \tag{7.2.18}$$

which is the desired result. □

Lemma 7.15. *Let $\{v_n\}$ be a sequence of $L^p(0,T;W_0^{1,p}(\Omega))-$functions such that*

$$\int_Q |\nabla T_k(v_n)|^p dxdt\le Ck, \tag{7.2.19}$$

for some positive constant C there exists a subsequence, still denoted by v_n, and a measurable function v such that v_n converges to v almost everywhere in Q.

Proof. See [**Pe1**], Theorem 6.1, Step. 2. □

7.3. Sketch of the Proof of Theorem 7.2

We will follow [**DMOP**] when dealing with nonlinear elliptic equations with measure data. Since the operator is monotone, there exists a unique solution u in W (this result is well known and is a consequence of [**LL**]) of the following nonlinear parabolic problem

$$\begin{cases} u_t-\text{div}(a(t,x,\nabla u))+|u|^{q-1}u=\mu & \text{in } (0,T)\times\Omega,\\ u(t,x)=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x)=u_0 & \text{in } \Omega,\end{cases}$$

in the sense that

$$\int_0^T\langle u_t,\varphi\rangle dt+\int_Q a(t,x,\nabla u)\cdot\nabla\varphi\ dxdt+\int_Q|u|^{q-1}u\varphi\ dxdt=\int_Q fd\mu, \tag{7.3.1}$$

for every φ in $S^p(Q)\cap L^\infty(Q)$ and for $\varphi=u$. So that $|u|^{q+1}$ (and $|u|^{q-1}u$) belong to $L^1(Q)$.

Step. 1 A priori estimates. We can choose $T_k(u_n)$ as a test function in the weak formulation of (7.1.3). We get, using (7.2.2) – (7.2.4), and the boundedness of (g_n) in $L^1(Q)$

$$\int_0^T\langle (u_n)_t,T_k(u_n)\rangle dt+\alpha\int_Q|\nabla T_k(u_n)|^p dxdt+\int_Q|u_n|^{q-1}u_nT_k(u_n)dxdt\le Ck, \tag{7.3.2}$$

for some positive constant C. Dropping the first two terms of the left hand side of the preceding inequality, we have

$$k\int_{\{|u_n|\geq k\}}|u_n|^q dxdt \leq \int_Q |u_n|^{q-1}|u_n||T_k(u_n)|dxdt \leq Ck,$$

so that

$$\int_{\{|u_n|\geq k\}}|u_n|^{q-1}|u_n|dx \leq C. \tag{7.3.3}$$

This implies

$$|k|^q \text{meas}\{|u_n|\geq k\} \leq k|k|^{q-1}\text{meas}\{|u_n|\geq k\} \leq C$$

and so $\{u_n\}$ is bounded in $\mathcal{M}^q(Q)$. Furthermore

$$\int_{\{|u_n|<k\}}|u_n|^{q-1}|u_n|dxdt \leq |k|^q\text{meas}(Q),$$

and so, using (7.3.3)

$$|u_n|^q \text{ is bounded in } L^1(Q). \tag{7.3.4}$$

The boundedness of u_n in $\mathcal{M}^q(Q)$, and Lemma 7.14, which can be applied since (7.3.2) also implies that

$$\int_Q |\nabla T_k(u_n)|^p dxdt \leq Ck \tag{7.3.5}$$

yields

$$\{|\nabla u_n|^{p-1}\} \text{ is bounded in } L^\sigma(Q) \text{ with } \sigma < \frac{pq}{(q+1)(p-1)}. \tag{7.3.6}$$

Moreover, equation (7.3.5) implies that $(T_k(u_n))$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, so that, by the weak lower semi-continuity of the norm, $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$, and thus u has a gradient ∇u in a suitable sense. As far the gradients of u_n, we remark that, u_n is the solution of the equation $u_t - \text{div}(a(t,x,\nabla u_n)) = f_n^\oplus - f_n^\ominus + g_n - |u_n|^{q-1}u_n$, and that the right hand side is bounded in $L^1(Q)$ by (7.1.10) and (7.3.4). By a result in **[BDGO]**, this implies that, up to subsequences,

$$\nabla u_n \text{ converges almost everywhere to } \nabla u. \tag{7.3.7}$$

From now on, we will suppose to have already extracted from u_n a subsequence (which we still denote by u_n), with the properties we have proved before. By (7.3.7) we have also

$$|\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \text{ strongly in } (L^\sigma(Q))^N, \tag{7.3.8}$$

we can apply Vitali's theorem, and we get $|\nabla u_n|^{p-1}\in L^\sigma(Q)$ and

$$\int_Q |\nabla u_n|^{p-1}dxdt \to \int_Q |\nabla u_n|^{p-1}dxdt. \tag{7.3.9}$$

Observing that, by assumption (7.2.3) on a, the argument above shows also that

$$a(t,x,\nabla u_n) \to a(t,x,\nabla u) \text{ strongly in } (L^\sigma(Q))^N, \tag{7.3.10}$$

for evry $\sigma < \frac{pq}{(q+1)(p-1)}$, the last convergence is also available in $L^1(Q)$.

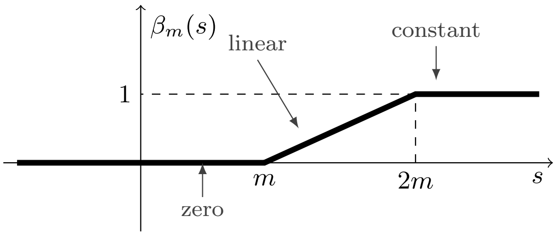
Step. 2 Energies estimates. Let $\psi_\delta = \psi_\delta^+ - \psi_\delta^-$, where ψ_δ^+ and ψ_δ^- are as in Lemma 7.10. Then

$$\int_{\{u_n>2m\}}|u_n|^q(1-\psi_\delta)dxdt = \omega(n,m,\delta), \tag{7.3.11}$$

and

$$\int_{\{u_n<-2m\}}|u_n|^q(1-\psi_\delta)dxdt = \omega(n,m,\delta). \tag{7.3.12}$$

We will only prove (7.3.11), since the proof of (7.3.12) is identical. We choose $\beta_m(u_n)(1-\psi_\delta)$ as test function in the weak formulation of (7.3.1), where $\beta_m(s)$ is defined as



$$\beta_m(s) = \begin{cases} 0 & \text{if } s \le m, \\ \frac{s}{m} - 1 & \text{if } m < s \le 2m, \\ 1 & \text{if } s > 2m, \end{cases} \tag{7.3.13}$$

FIGURE 24. The function $\beta_m(s)$

we obtain, using the fact that the derivation of $\beta_m(s)$ is different from zero only where $m < s < 2m$,

$$\begin{aligned}
&\int_0^T \langle (u_n)_t, \beta_m(u_n)(1-\psi_\delta)\rangle dt && (A)\\
&+\frac{1}{m}\int_{\{m<u_n\le 2m\}} a(t,x,\nabla u_n)\cdot\nabla u_n(1-\psi_\delta)dxdt && (B)\\
&-\int_Q a(t,x,\nabla u)\cdot\nabla\psi_\delta\beta_m(u_n)dxdt && (C)\\
&+\int_Q |u_n|^{q-1}u_n\beta_m(u_n)(1-\psi_\delta)dxdt && (D)\\
&=\int_Q f_n^{\oplus}\beta_m(u_n)(1-\psi_\delta)dxdt && (E)\\
&-\int_Q f_n^{\ominus}\beta_m(u_n)(1-\psi_\delta)dxdt && (F)\\
&+\int_Q g_n\beta_m(u_n)(1-\psi_\delta)dxdt && (G)
\end{aligned}$$

we have, by (7.3.4), by Egorov theorem, and since $\beta_m(u_n)$ converges to $\beta_m(u)$ almost everywhere in Q and in the weak–* topology of $L^\infty(Q)$

$$-(C) = \int_Q a(t,x,\nabla u)\cdot\nabla\psi_\delta\beta_m(u)dxdt + \omega(n) = \omega(n,m),$$

and the last passage is due to the fact that $\beta_m(u)$ converges to zero in the weak–* topology of $L^\infty(Q)$ as m tends to infinity. For the same reason, we have

$$(G) = \omega(n,m).$$

Finally, by (7.2.9) – (7.2.11)

$$\begin{aligned}
(E) &\le \int_Q f_n^{\oplus}(1-\psi_\delta)dxdt = \int_Q f_n^{\oplus}(1-\psi_\delta^+)dxdt + \int_Q f_n^{\oplus}\psi_\delta^-\,dxdt\\
&= \int_Q (1-\psi_\delta^+)d\lambda^+ + \int_Q \psi_\delta^-\,d\lambda^+ + \omega(n)\\
&= \omega(n,\delta).
\end{aligned}$$

Since (B) and $-(F)$ are nonnegative, and since

$$(D) \ge -\int_{\{u_n>2m\}} |u_n|^q(1-\psi_\delta)dxdt,$$

and since

$$(A) = \int_Q B_m(u_n)\psi_\delta)_t + \int_\Omega B_m(u_n)(T) \ge \omega(n,m)$$

(which B_m is the primitive of β_m). We have the result (7.3.11).

Step. 3 Passing to the limit. We are now ready to conclude the proof of Theorem 7.2, showing that u is the entropy solution of (7.1.4) with datum g: Let φ be a function in $S^p \cap L^\infty(Q)$, $M = \|\varphi\|_{L^\infty(Q)}$, $k > 0$ and choosing $T_k(u_n - \varphi)(1 - \psi_{\delta\eta})h_m(u_n)$ (with $\psi_{\delta\eta} = \psi_\delta^+\psi_\eta^+ + \psi_\delta^-\psi_\eta^-$ and $h_m(s) = 0$ if $|s| > 2m$, $h_m(s) = 2 - \frac{|s|}{m}$ if $m < |s| \leq 2m$, and 1 if $|s| \leq m$) as test function in the weak formulation of (7.1.3), we get

$$\begin{aligned}
&\int_0^T \langle (u_n)_t, T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)\rangle dt && (A)\\
&+ \int_Q a(t,x,\nabla u_n)\cdot \nabla T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)dxdt && (B)\\
&- \int_Q a(t,x,\nabla u)\cdot \nabla \psi_{\delta\eta} T_k(u_n-\varphi)h_m(u_n)dxdt && (C)\\
&+ \int_Q |u_n|^{q-1}u_n T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)dxdt && (D)\\
&= \int_Q f_n^{\oplus} T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)dxdt && (E)\\
&- \int_Q f_n^{\ominus} T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)dxdt && (F)\\
&+ \int_Q g_n T_k(u_n-\varphi)(1-\psi_{\delta\eta})h_m(u_n)dxdt && (G)\\
&- \frac{1}{m}\int_{\{m<u_n\leq 2m\}} a(t,x,\nabla u_n)\cdot \nabla u_n(1-\psi_{\delta\eta})T_k(u_n-\varphi)dxdt && (H)\\
&+ \frac{1}{m}\int_{\{-2m\leq u_n<-m\}} a(t,x,\nabla u_n)\cdot \nabla u_n(1-\psi_{\delta\eta})T_k(u_n-\varphi)dxdt && (I)
\end{aligned}$$

using (7.3.11), one has the convergence of $a(t,x,\nabla u_n)$ to $a(t,x,\nabla u)$ in $L^\sigma(Q)$. Thus using (7.2.9) we get

$$-(C) = \int_Q a(t,x,\nabla u)\cdot \nabla\psi_\delta T_k(u-\varphi)dxdt + \omega(n) = \omega(n,\delta,\eta),$$

using (7.2.10) and (7.2.11), we obtain

$$|(E)| + |(F)| \leq k\int_Q (f_n^{\oplus} + f_n^{\ominus})(1-\psi_{\delta\eta})dxdt = \omega(x,\delta,\eta).$$

It is easy to see that

$$(G) = \int_Q gT_k(u-\varphi)dxdt + \omega(n,\delta,\eta)$$

so that we only have to deal with (A), (B) and (C). Let $m > k + M$ be fixed. We then have

$$\begin{aligned}
(D) = &\int_{\{-2m\leq u_n\leq 2m\}} |u_n|^{q-1}u_n T_k(u_n-\varphi)(1-\psi_{\delta\eta})dxdt && (H)\\
&+ \int_{\{u_n>2m\}} |u_n|^{q-1}u_n k(1-\psi_{\delta\eta})dxdt && (I)\\
&+ \int_{\{u_n<-2m\}} |u_n|^{q-1}|u_n| k(1-\psi_{\delta\eta})dxdt && (J)
\end{aligned}$$

It is easily seen that (recall that $|u|^{q-1}u \in L^1(Q)$)

$$\begin{aligned}
(H) &= \int_{\{-2m\leq u\leq 2m\}} |u|^{q-1}uT_k(u-\varphi)(1-\psi_{\delta\eta})dxdt + \omega(n)\\
&= \int_Q |u|^{q-1}uT_k(u-\varphi)(1-\psi_{\delta\eta}) + \omega(n,m)\\
&= \int_Q |u|^{q-1}uT_k(u-\varphi)dxdt + \omega(n,m,\delta)
\end{aligned}$$

we then have, by (7.3.11),

$$(I) = k \int_{\{u_n > 2m\}} |u_n|^{q-1} u_n (1 - \psi_{\delta\eta}) dxdt = \omega(n, m, \delta, \eta),$$

and, by (7.3.12),

$$(J) = k \int_{\{u_n < -2m\}} |u_n|^{q-1} |u_n| (1 - \psi_{\delta\eta}) dxdt = \omega(n, m, \delta, \eta),$$

so that

$$(D) = \int_Q |u|^{q-1} u T_k(u - \varphi) dxdt + \omega(n, \delta, \eta).$$

Finally, we have

$$\begin{aligned}(B) = &\int_Q [a(t, x, \nabla u_n) - a(t, x, \nabla \varphi)] \cdot \nabla T_k(u_n - \varphi)(1 - \psi_{\delta\eta}) dxdt &&(K)\\ &+ \int_Q a(t, x, \nabla \varphi) \cdot \nabla T_k(u_n - \varphi)(1 - \psi_{\delta\eta}) dxdt. &&(L)\end{aligned}$$

Since the integral function in (K) is nonnegative, and converges almost everywhere in Q to $[a(t, x, \nabla u) - a(t, x, \nabla \varphi)] \cdot \nabla T_k(u - \varphi)$, as n tends to infinity and then δ tends to zero, Fatou's lemma implies

$$\int_Q [a(t, x, \nabla u) - a(t, x, \nabla \varphi)] \cdot \nabla T_k(u - \varphi) dxdt \leq \liminf_{\delta \to 0^+} \liminf_{\eta \to 0^+} \liminf_{n \to \infty} (K).$$

Moreover, since $a(t, x, \nabla \varphi)$ belongs to $(L^{p'}(Q))^N$, we have

$$(L) = \int_Q a(t, x, \nabla \varphi) \cdot \nabla T_k(u - \varphi) dxdt = \omega(n, \delta, \eta),$$

so that, putting together the results for (K) and (L), we have

$$\int_Q a(t, x, \nabla u) \cdot \nabla T_k(u - \varphi) dxdt \leq \liminf_{\delta \to 0^+} \liminf_{\eta \to 0^+} \liminf_{n \to \infty} (B)$$

Summing up the results we have obtained so far, we have

$$\int_Q a(t, x, \nabla u) \cdot \nabla T_k(u - \varphi) dxdt + \int_Q |u|^{q-1} u T_k(u - \varphi) dxdt \leq \int_Q g T_k(u - \varphi) dxdt,$$

and so u is the entropy solution of (7.1.4). Observe that, thanks to the uniqueness of entropy solution, the solution u does not depend on the subsequences we have extracted, then the whole sequence u_n converges to u. To conclude the proof of the theorem, it only remains to prove (7.1.5). In order to do this, we choose a test function $\varphi \in C_c^\infty(Q)$ in the weak formulation of (7.1.3), we get

$$\int_Q a(t, x, \nabla u_n) \cdot \nabla \varphi dxdt + \int_Q |u_n|^{q-1} u_n \varphi dxdt = \int_Q (f_n + g_n) \varphi dxdt.$$

Thanks to (7.3.10), and the assumptions on f_n and g_n, we have

$$\int_Q |u_n|^{q-1} u_n \varphi dxdt = -\int_Q a(t, x, \nabla u) \varphi dxdt + \int_Q g \varphi dxdt + \int_Q \varphi d\lambda + \omega(n)$$

since the entropy solution of (7.1.4) is also a distributional solution of the some problem, we have for the some φ,

$$\int_Q a(t, x, \nabla u) \cdot \nabla \varphi dxdt + \int_Q |u|^{q-1} u \varphi dxdt = \int_Q g \varphi dxdt,$$

and so we have proved that (7.1.5) holds for every φ in $C_c^\infty(Q)$. Since $|u_n|^{q-1} u_n$ is bounded in $L^1(Q)$, equation (7.1.5) can then be extended by density to the functions in $C_c^0(Q)$.

7.4. Proof of the main result

Now, let us come back to the proof of the theorem 7.7.

Step. 1 A priori estimates. Taking $v = g_2$ in the equation (7.1.7), we have

$$
\begin{aligned}
&\int_0^T \langle (u_n - g_2)_t, g_2 - u_n \rangle dt - \int_Q a(t,x,\nabla u_n)\cdot\nabla(g_2 - u_n)dxdt \\
&\quad \geq \int_Q g_1(g_2 - u_n)dxdt - \int_Q G\cdot\nabla(g_2 - u_n)dxdt + \int_Q f_n(g_2 - u_n)dxdt
\end{aligned}
\tag{7.4.1}
$$

from which it follows by (7.2.2)

$$
\begin{aligned}
&\int_\Omega \left[\frac{(u_n - g_2)^2}{2}\right]_0^t dx + \alpha\int_0^t\int_\Omega |\nabla u_n|^p dx - \int_0^t\int_\Omega a(t,x,\nabla u_n)\cdot\nabla g_2 dxdt \\
&\leq \int_\Omega g_1(u_n - g_2)dxdt - \int_\Omega G\cdot\nabla(u_n - g_2)dxdt + \int_Q f_n^{\oplus}(u_n - g_2)dxdt - \int_Q f_n^{\ominus}(u_n - g_2)dxdt.
\end{aligned}
\tag{7.4.2}
$$

Recall that $(u_n - g_2)(0) = u_n(0) = u_0^n$ and using Young's inequality, this gives

$$
\begin{aligned}
&\int_\Omega \frac{(u_n - g_2)^2(t)}{2}dx - \int_\Omega \frac{u_n^2(0)}{2}dx + \alpha\int_0^t\int_\Omega |\nabla u_n|^p dxdt \leq \|g_1\|_{L^1(Q)} + \int_0^t\int_\Omega G\cdot\nabla u_n dxdt \\
&+ \int_0^t\int_\Omega G\cdot\nabla g_2 dxdt + \int_0^t\int_\Omega a(t,x,\nabla u_n)\cdot\nabla g_2 dxdt + \|f_n^+\|_{L^1(Q)} - \|f_n^-\|_{L^1(Q)}.
\end{aligned}
$$

Using again Young's inequality and assumption (7.2.2), we get

$$
\begin{aligned}
&\int_\Omega \frac{(u_n - g_2)^2(t)}{2}dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla u_n|^p dxdt \leq \|g\|_{L^1(Q)} + C\int_0^t\int_\Omega |G|^{p'}dxdt + C\int_Q |\nabla g_2|^p dxdt \\
&+ C\int_Q |b(t,x)|^p dxdt + C\|u_0\|^2_{L^2(\Omega)} + \|f_n\|_{L^1(Q)}.
\end{aligned}
$$

From now on C denotes a constant that may change from one line to another. Then,

$$
\begin{aligned}
&\int_\Omega \frac{(u_n - g_2)^2(t)}{2}dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla u_n|^p dxdt \\
&\leq C\ (\|g\|_{L^1(Q)} + \|G\|_{(L^{p'}(Q))^N} + \|g_2\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|f_n\|_{L^1(Q)} + \|u_0\|^2_{L^2(\Omega)}).
\end{aligned}
\tag{7.4.3}
$$

We obtain

$$
\int_\Omega \frac{(u_n - g_2)^2(t)}{2}dx \leq C \quad \forall t \in (0,T),
\tag{7.4.4}
$$

which implies the estimate of $u_n - g_2$ in $L^\infty(0,T;L^2(\Omega))$, and also

$$
\int_Q |\nabla u_n|^p dxdt \leq C,
\tag{7.4.5}
$$

which yields that u_n and $u_n - g_2$ are bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ (recall that g_2 is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). Thus, up to a subsequence, still denoted by u_n and $u_n - g_2$, u_n converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to some finite w which is easily seen to belong to K.

Step. 2 Near the support of λ. We have

$$
\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n - g_2)\psi_\delta^+ dxdt = \omega(n,\delta),
\tag{7.4.6}
$$

and

$$
\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n - g_2)\psi_\delta^- dxdt = \omega(n,\delta).
\tag{7.4.7}
$$

Moreover, Let $v = \psi_\delta^+ + (u_n - g_2)(1 - \psi_\delta^+)$; it is easy to see that v belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, since both ψ_δ^+, u_n and g_2 belongs to $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. Then by definition of ψ_δ^+ (i.e., $0 \le \psi_\delta^+ \le 1$), v belongs to K. On the other hand, Taking v in (7.1.7), we have, since $v - u_n = (1 - u_n + g_2)\psi_\delta^+$,

$$
\begin{aligned}
&\int_0^T \langle (u_n - g_2)_t, (1 - u_n + g_2)\psi_\delta^+ \rangle dt && (A)\\
&- \int_Q a(t,x,\nabla u_n) \cdot \nabla(u_n - g_2)\psi_\delta^+ dxdt && (B)\\
&+ \int_Q a(t,x,\nabla u_n) \cdot \nabla\psi_\delta^+ (1 - u_n + g_2) dxdt && (C)\\
&\ge \int_Q g_1(1 - u_n + g_2)\psi_\delta^+ dxdt && (D)\\
&- \int_Q G \cdot \nabla(u_n - g_2)\psi_\delta^+ dxdt && (E)\\
&+ \int_Q G \cdot \nabla\psi_\delta^+ (1 - u_n + g_2) dxdt && (F)\\
&+ \int_Q f_n^+ (1 - u_n + g_2)\psi_\delta^+ dxdt && (G)\\
&- \int_Q f_n^- (1 - u_n + g_2)\psi_\delta^+ dxdt && (H)
\end{aligned}
$$

Now, since (u_n) is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, the sequence $(a(t,x,\nabla u_n))$ is bounded in $L^{p'}(Q)$ by (7.2.3), we have that for a subsequence of u_n, denoted equal, $a(t,x,\nabla u_n)$ converges weakly to some σ in $(L^{p'}(Q))^N$. Thus, since $1 - u_n + g_2$ converges weakly in $L^\infty(Q)$ to $1 - w + g_2$, we get

$$(C) = \int_Q \sigma \cdot \nabla\psi_\delta^+ (1 - w + g_2) dxdt + \omega(n) = \omega(n,\delta),$$

since ψ_δ^+ converges strongly to zero in $L^p(0,T;W_0^{1,p}(\Omega))$. On the other hand

$$(D) = \int_\Omega g_1(1 - w + g_2)\psi_\delta^+ dxdt + \omega(n) = \omega(n,\delta),$$

since ψ_δ^+ converges to zero in the weak–* topology of $L^\infty(Q)$. Therefore

$$(E) = \int_Q G \cdot \nabla(w - g_2)\psi_\delta^+ dxdt + \omega(n) = \omega(n,\delta),$$

again because ψ_δ^+ converges to zero in the weak–* topology of $L^\infty(Q)$, and

$$(F) = \int_Q G \cdot \nabla\psi_\delta^+ (1 - w + g_2) dxdt + \omega(n) = \omega(n,\delta),$$

due to the strong convergence to zero of ψ_δ^+ in $L^p(0,T;W_0^{1,p}(\Omega))$. Moreover, we have

$$(H) \le 2\int_Q f_n^\ominus \psi_\delta^- dxdt = 2\int_Q \psi_\delta^+ d\lambda^- + \omega(n) = \omega(n,\delta).$$

Now let us see how to pass to the limit in (A)

$$
\begin{aligned}
-(A) &= -\left[\int_0^T (u_n - g_2)_t, (1 - u_n + g_2)\psi_\delta^t\right]\\
&= \left[\int_0^T (u_n - g_2)_t, \psi_\delta^+ \rangle dt - \int_0^T \langle (u_n - g_2)_t, u_n - g_2 \rangle dt\right]\\
&= \int_0^T \langle (u_n - g_2)(\psi_\delta^+)_t dxdt + \left[\frac{(u_n - g_2)^2}{2}\right]_0^T\\
&\ge \omega(n,\delta).
\end{aligned}
$$

Using the fact that (F) is non-negative, we get

$$-(B)=\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n-g_2)\psi_\delta^+dx=\omega(n,\delta),$$

that is (7.4.6), in particular, formula (7.4.7) is obtained in the same way by taking $v=-\psi_\delta^-+(u_n-g_2)(1-\psi_\delta^-)$ (which still belong to K) in the equation (7.1.7).

Step. 3 Far from the support of λ. We have

$$\int_Q a(t,x,\nabla u_n)-a(t,x,\nabla u)\cdot\nabla(u_n-u)(1-\psi_\delta^+-\psi_\delta^-)dxdt=\omega(n,\delta). \tag{7.4.8}$$

Define, as in the proof of Theorem 7.2, $\psi_\delta=\psi_\delta^+-\psi_\delta^-$, and choose $v=(u_n-g_2)\psi_\delta+(u_n-g_2)(1-\psi_\delta)$ as a test function in (7.1.9). Observe that both test functions belongs to K, since the supports of ψ_δ^+ and ψ_δ^- are disjoint. We get

$$\begin{aligned}&\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n-u)(1-\psi_\delta)dxdt-\int_Q a(t,x,\nabla u_n)\cdot\nabla\psi_\delta(u_n-u)dxdt\\
&\le\int_Q g_1(u_n-u)(1-\psi_\delta)dxdt+\int_Q G\cdot\nabla(u_n-u)(1-\psi_\delta)dxdt\\
&-\int_Q G\cdot\nabla\psi_\delta(u_n-u)dxdt+\int_Q f_n^\oplus(u_n-u)(1-\psi_\delta)dxdt\\
&-\int_Q f_n^\ominus(u_n-u)(1-\psi_\delta)dxdt,\end{aligned}$$

and

$$\begin{aligned}&-\int_Q a(t,x,\nabla u)\cdot\nabla(u_n-u)(1-\psi_\delta)dxdt+\int_Q a(t,x,\nabla u)\cdot\nabla\psi_\delta(u_n-u)dxdt\\
&\le-\int_Q g_1(u_n-u)(1-\psi_\delta)dxdt-\int_Q G\cdot\nabla(u_n-u)(1-\psi_\delta)dxdt\\
&+\int_Q G\cdot\nabla\psi_\delta(u_n-u)dxdt.\end{aligned}$$

Summing up, we get

$$\begin{aligned}&\int_Q(a(t,x,\nabla u_n)-a(t,x,\nabla u))\cdot\nabla(u_n-u)(1-\psi_\delta)dxdt && (A)\\
&\le\int_Q a(t,x,\nabla u_n)-a(t,x,\nabla u)\cdot\nabla\psi_\delta(u_n-u)dxdt && (B)\\
&+\int_Q f_n^\oplus(u_n-u)(1-\psi_\delta)dxdt && (C)\\
&-\int_Q f_n^\ominus(u_n-u)\cdot(1-\psi_\delta)dxdt\end{aligned}$$

Using the boundedness of $(a(t,x,\nabla u_n))$ in $L^{p'}(Q)$, and reasoning as in *Step. 2*, it is easy to see that

$$(B)=\int_Q(a(t,x,\nabla u_n)-a(t,x,\nabla u))\cdot\nabla\psi_\delta dxdt=\omega(n,\delta).$$

On the other hand, we have

$$\begin{aligned}|(C)|&\le 2\int_Q f_n^\oplus(1-\psi_\delta)dxdt\\
&=2\int_Q(1-\psi_\delta^+-\psi_\delta^-)d\lambda^++\omega(n)\\
&\le 2\int_Q(1-\psi_\delta^+)d\lambda^++\int_Q\psi_\delta^-d\lambda^++\omega(n)\\
&=\omega(n,\delta)\end{aligned}$$

The same technique implies

$$(A) = \omega(n, \delta),$$

that is is (7.4.8).

Step. 4 Passing to the limit. We have u_n converges strongly in $L^p(0,T; W_0^{1,p}(\Omega))$. This will be true, thanks to the assumption on a, and to a result in [**Brow**], if we prove that

$$\int_Q (a(t,x,\nabla u_n)) - a(t,x,\nabla u) \cdot \nabla(u_n - u)dx = \omega(n). \tag{7.4.9}$$

In order to prove (7.4.9), we can use the results of *Step. 2* and *Step. 3* , decomposing the integral by means of the function ψ_δ^+ and ψ_δ^-. Then, by (7.4.8), we only have to deal with

$$\int_Q (a(t,x,\nabla u_n)) - a(t,x,\nabla u)) \cdot \nabla(u_n - u)\psi_\delta dxdt,$$

where, as before, $\psi_\delta = \psi_\delta^+ + \psi_\delta^-$. The integral can be decomposed in the some four terms,

$$\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n - g_2)\psi_\delta dx, \quad \int_Q a(t,x,\nabla u)\cdot\nabla(u - g_2)\psi_\delta dx,$$

$$-\int_Q a(t,x,\nabla u)\cdot\nabla(u_n - g_2)\psi_\delta dx, \quad -\int_Q a(t,x,\nabla u_n)\cdot\nabla(u_n - g_2)\psi_\delta dx.$$

The first one is an $\omega(n,\delta)$, by (7.4.6) and (7.4.7); the second one is an $\omega(\delta)$, since $\psi_\delta^+ + \psi_\delta^-$ converges to zero in the weak–* topology of $L^\infty(Q)$, and u, g_2 belongs to $L^p(0,T; W_0^{1,p}(\Omega))$; for the third term, we have

$$-\int_Q a(t,x,\nabla u)\cdot\nabla(u_n - g_2)\psi_\delta dx = -\int_Q a(t,x,\nabla u)\cdot\nabla(w - g_2)dx + \omega(n) = \omega(n,\delta),$$

always because ψ_δ converges to zero in the weak–* topology of $L^\infty(Q)$. Finally, for the fourth term we have, by Hölder's inequality, by (7.2.3), and by the boundedness of u_n and g_2 in $L^p(0,T; W_0^{1,p}(\Omega))$

$$\left|\int_Q a(t,x,\nabla u_n)\cdot\nabla(u - g_2)\psi_\delta dxdt\right| \le C\left(\int_Q |\nabla u|^p\psi_\delta dx\right)^{\frac{1}{p}} = \omega(\delta).$$

Since u, g_2 belongs to $L^p(0,T; W_0^{1,p}(\Omega))$ and ψ_δ converges to zero in the weak–* topology of $L^\infty(Q)$. This proves that u_n converges to u strongly in $L^p(0,T; W_0^{1,p}(\Omega))$. Since the limit is independent of the subsequence extracted, the whole sequence u_n converges to u, and so the proof of the theorem 7.7 is finished.

CHAPTER 8

Nonlinear parabolic problems with blowing up coefficients and general measure data

Nonlinear diffusion equations, as an important class of parabolic equations, come from a variety of diffusion phenomena appeared widely in nature. They are suggested as mathematical models of physical problems in many fields such as energy dissipation, Navier-Stokes flow, turbulent transition, viscosity and incompressible fluid mechanics. In many cases the equations possess a velocity field w and a pressure of the fluid p. Comparing to the Reynolds number Re (a positive constant linked to the flow), such equations, to a certain value of Re, reflect even more exactly the physical reality of the flow. For example, when this value is high enough, the flow becomes unstable and turbulent structures involving both the velocity field and pressure may appear. the numerical solutions of such equations is an arduous task due to the large number of nodes of an appropriate mesh (the interest reader may refer to the papers **[Fr1, Fr2]** for more applications). This Chapter is devoted to the study of some of this evolution problems whose model

$$(8.0.1)\qquad \begin{cases} u_t - \operatorname{div}(d(u)Du) = \mu & \text{in } (0,T)\times\Omega,\\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x) = u_0 & \text{in } \Omega,\end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $T>0$, $Q=(0,T)\times\Omega$, $d(s)=(d_i(s))_{i=1}^N$ is a diagonal matrix, such that the coefficients $d_i(s)$ are continuous on an interval $]-\infty,m[$ of $\mathbb{R}$ ($m>0$) with values in $\mathbb{R}^+\cup\{+\infty\}$, there exists $\alpha>0$ such that $d_i(s)\geq\alpha$ for all $s\leq m$ and all $i\in\{1,\cdots,N\}$, there exists an index p such that $\lim\limits_{s\to m^-} d_p(s)=+\infty$, $u_0\in L^1(\Omega)$ with $u_0<m$ a.e. in Ω and μ is a general measure on Q with bounded total variation. Due to the presence of the character $d(u)Du$, problems (8.0.1) enters the class of parabolic problems with blowing up coefficients for which there exists a larger number of references. Among them **[BGR, BR1, BR2, Fr1, Fr2, R, VG1, VG2, VG3]**, let us mention that a priori estimates do not lead in general to the existence of a weak solution, because there are mainly two difficulties: one consists in defining the field $d(u)Du$ on the subset $\{(t,x)\in Q\colon u(t,x)=m\}$ of Q, since, on this set, $d_p(u)=+\infty$ due to the singular behaviour of $d(s)$ as s tends to m, we can not set in general $d_p(u)\frac{\partial u}{\partial x_p}=0$ on $\{(t,x)\in Q : u(t,x)=m\}$. A natural technique to define $d(u)Du$ is to exploit the *a priori* estimates that can be derived on approximate problems, the second difficulty is the consideration of L^1 initial datum and general measure data. So that distributional solutions could not be expected. Indeed, since a few years, the framework of renormalized solutions has proved to be a powerful approach to study this class of partial differential equations with L^1 and measure data. As far as a reader that is not familiar with this notion is concerned, just recall that it consists of multiplying the pointwise equation (8.0.1) by a function of the type $S(u)$, where S is any smooth function such that the support of S is compact. We address then problems (8.0.1) in this setting and we prove that if d and μ satisfies some assumptions, it is well-posed. In particular, we establish a new existence result which extends in possibly different directions previous results dealing with this question. Note that, in stationary case where $\mu\in L^2(\Omega)$, it is well known that the existence of solutions for problems (8.0.1) using an estimate of $d(u_n)Du_n$ was proved in **[BR1]** (see also **[BR2]**) where the authors gave two formulations of problem of type (8.0.1), both of them using a sort of decoupling behavior of the solution on the subset $\{u<m\}$ and on the subset $\{u=m\}$. Let recall that, in **[Or]**, Orsina has analyzed the case when μ is a bounded Radon measure on Ω , but his model leads to $L^\infty-$estimates for the solutions. Moreover, a similar notions certainly closer to the one used in **[BR2, Or]** was used to get existence and uniqueness of solutions for some nonlinear elliptic problems encountered in physical models, such as the turbulence models derived from the

Navier-Stokes equations but with right-hand side $\mu \in L^1(\Omega)$ and also for the parabolic case with $\mu \in L^1(Q)$ [**VG2, VG3**]. Indeed another type of diffusion-problems can be adopted when $d(u) = d(u) + A(u)$ is a diffusion matrix that has a non-controlled growth with respect to the unknown u and that has a diagonal coefficient $d_p(u) + A_{pp}(u)$ that blows up for the finite value m of u. Let us just mention that this type of behavior for diffusion matrices are encountered in physical setting where an internal variable u is constrained to remain smaller than m [**Fr1, Fr2**]. For the stationary cases with singular matrices with respect to the unknown, more precisely the case $d(u) = A(x, u)$ where $A(x, u)$ is a Carathéodory function from $\Omega \times (-\infty, m)$ into $\mathbb{R}_s^{N\times N}$ (the set of $N \times N$ symmetric matrices) and the case $d(u) = A(t, x, u)$ where $A(t, x, u)$ is a Carathéodory matrix defined on $(0, T) \times \Omega \times (-\infty, m)$ not diagonal and blow up (uniformly with respect to (t, x)) as $s \to m^-$ was proved in [**BGR**] for L^1- integrable data and in [**ZR**] for diagonal field and diffuse measures which does not charges sets of zero $2-$capacity. For instance if $\text{div}(d(u)Du)$ is replaced by the $p-$Lalpacian operator, the existence of renormalized was done in Chapter 1. An interesting and complete discussion of this point can be found in Chapter 4. A powerful method to obtain extensions for more general nonlinear operators in divergence form is the strong approximation of measures which dates back to [**PPP1**] and which has been extensively developed in [**PPP2**] and then applied in a recent series of papers (see Chapter 5). In these works, the authors perform a complete results based on a decomposition theorem for diffuse measure (Theorem 8.5). However, this decomposition can not be easily used for problems of type (8.0.1) with absorption terms (at least with reasonable assumptions on the time character g derived from the decomposition of μ). On the other hand, the right-hand side of (8.0.1) suggests that it should be more natural to include the doubly cut-off functions to deal with general, possibly, singular measures [**Pe1**]. We show in the present paper that this approach (i.e. doubling cut-off functions) still works even with problems with vector field $d(s)$ assuming a specific assumptions on the continuous coefficients $d_i(s)$. Of course, the hardest task that we face in handling the zones where μ_s (μ_s, are measures concentrated on a sets of zero $2-$capacity) are concentrated, which we treat with an idea inspired by an argument used in [**DMOP**]. In particular, the technical tools that we use here which allow to deal with these type of problems, is the notion of parabolic capacity and equi-diffuse measures. Moreover, it is very important to remark that the proof of existence works without the decomposition of the measure data so that it can be applied also to porous-medium parabolic problems (see Chapter 6). By contrast, a capacity estimate of u is needed in the proof of the existence result (Lemma 8.4). Here the main argument relies on approximation properties of the measure, with respect to nonlinear potential of the data and of the truncated potential, which is proved to be element of $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ (see Theorem 8.7). The tools needed for obtaining this kind of result have been widely developed for diffuse measures μ making use of a particular convolution-regularization introduced in [**BP**] (see also [**MP**]). In this Chapter, we use generalized version of this convolution result which uses a mean regularization together with the singular part of μ, the existence of renormalized solution is then obtained by passing to the limit in the difference of diffuse and singular terms, and this is where we use the assumption that μ is equi-diffuse. A reader who is willing to accept this pointwise convergence without assuming the boundedness of g in $L^\infty(Q)$ in the decomposition of μ and without the strong convergence of truncates. This Chapter is organized as follows. In the next Section, we propose some tools, which will play a crucial role in our proof. Section 8.2 is devoted to the main assumptions and this will lead to introduce a new definition of renormalized solutions to the problem (8.0.1). The main result is based on approximate problems whose solutions satisfy the a priori estimates of Section 8.3. We end with the proof of the main result (Theorem 8.18) under more restrictive conditions on test functions.

8.1. Some preliminary results on parabolic 2−capacity

In the following, we denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the $\sigma-$algebra of Borelian subsets of Q equipped with the norm $\|\mu\|_{\mathcal{M}_b(Q)} = |\mu|(Q)$. The approach followed to define the capacity is in the same spirit as in [**P, DPP**].

Definition 8.1. Let us define $V = H_0^1(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^2(\Omega)}$, and

$$W = \{u \in L^2(0, T; H_0^1(\Omega)),\ u_t \in L^2(0, T; H^{-1}(\Omega))\}.$$

We will define the parabolic capacity using the space W

Definition 8.2. If $U \subset Q$ is an open set, we define the parabolic capacity of U as

$$\mathrm{cap}_2(U) = \inf\{\|u\|_W : u \in W,\ u \geq \chi_U \text{ almost everywhere in } Q\}, \tag{8.1.1}$$

(we will use the convention that $\inf \emptyset = +\infty$), then for any Borelian subset $B \subset Q$, the definition is extended by setting

$$\mathrm{cap}_2(B) = \inf\{\mathrm{cap}_2(U),\ U \text{ open subset of } Q,\ B \subset U\}.$$

Let us recall that a function u is called cap_2 quasi-continuous if for every $\epsilon > 0$ there exists an open set F_ϵ, with $\mathrm{cap}_2(F_\epsilon) \leq \epsilon$, and such that $u_{|(Q_{|F_\epsilon})}$ (the restriction of u to $Q_{|F_\epsilon}$) is continuous in $Q_{|F_\epsilon}$. As usual, a property will be said to hold cap_2 quasi-everywhere if it holds everywhere expect on a set of zero capacity.

Let us introduce some new notations: if F is a function of one real variable, then $\overline{F}$ will denote its primitive function, that is $\overline{F}(s) = \int_0^s F(r)dr$. We will indicate simply with S the space S as

$$S = \{u \in L^2(0,T;H_0^1(\Omega));\ u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)\},$$

endowed with its natural norm $\|u\|_S = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|u_t\|_{L^2(0,T;H^{-1}(\Omega))+L^1(Q)}$, and its subspace W_1 as

$$W_1 = \{z \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q),\ z_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)\},$$

endowed with its natural norm $\|\cdot\|_{L^2(0,T;H_0^1(\Omega))} + \|\cdot\|_{L^\infty(Q)} + \|\cdot\|_{L^2(0,T;H^{-1}(\Omega))+L^1(Q)}$. Therefore, thanks to the Young's inequality and to the fact that W_1 is continuously embedded in $C([0,T];L^1(\Omega))$ **[Po1]** we have

Proposition 8.3. *If u is cap_2 quasi-continuous and belong to W_1, then for all $k > 0$*

$$\mathrm{cap}_2(\{|u| > k\}) < \frac{C}{k}\max\{\|u\|_{W_1}^2\}. \tag{8.1.2}$$

Proof. See **[Pe1]**, Theorem 3 and Lemma 2. □

In particular, for solutions of parabolic equations we have a capacitary estimate on the level sets of u

Lemma 8.4. *Given $\mu \in \mathcal{M}_b(Q) \cap L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $d_i \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $i \in \{1,...,N\}$ and $u \in W$ be the (unique) weak solution of problem (8.0.1). Then*

$$\mathrm{cap}_2(\{|u| > k\}) \leq \frac{C}{k^{\frac{1}{2}}}, \quad \forall k \geq 1, \tag{8.1.3}$$

where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}_b(Q)}$, $\|u_0\|_{L^2(\Omega)}$.

Proof. See **[ZR]**, Theorem 2.3. □

In (8.1.3), u is identified with its cap_2 quasi-continuous representative, which exists since $u \in W$ **[DPP]** and the quantity $\mathrm{cap}_2(\{|u| > k\})$ is well-defined. In order to better specify the notion of measures in $\mathcal{M}_0(Q)$, we need then to detail the decomposition theorem for its elements.

Theorem 8.5. *Let $\mu \in \mathcal{M}_0(Q)$, then there exists (f, g, χ) such that $f \in L^1(Q)$, $g \in L^2(0,T;H_0^1(\Omega))$ and $\chi \in L^2(0,T;H^{-1}(\Omega))$ such that*

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_0^T \langle\chi,\varphi\rangle dt - \int_0^T \langle\varphi_t, g\rangle dt, \quad \forall \varphi \in C_c^\infty([0,T]\times\Omega), \tag{8.1.4}$$

and the triplet (f, g, χ) will be called a decomposition of μ.

Proof. See **[DPP]**, Theorem 2.28. □

The possibility that the above decomposition holds for some $g \in L^\infty(Q)$ has a special interest, as it was also pointed out in **[PPP1]** and in Chapter 5. In particular, one has the following counterpart

Proposition 8.6. *Assume $\mu \in \mathcal{M}(Q)$ satisfies (8.1.4), where $f \in L^1(Q)$, $g \in L^2(0,T;H_0^1(\Omega))$ and $\chi \in L^2(0,T;H^{-1}(\Omega))$. If $g \in L^\infty(Q)$, then μ is diffuse.*

Proof. See **[PPP2]**, Proposition 3.1. □

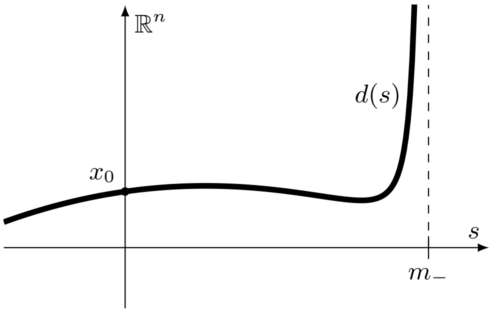


FIGURE 25. Blow up phenomenon

We will now state, thanks to what has been done in Theorem 8.1.6 and Proposition 8.6, an approximation result concerning elements of $\mathcal{M}_0(Q)$, which will allow us to obtain additional regularity results on the renormalized solutions of (8.0.1).

THEOREM 8.7. *Let $\mu \in \mathcal{M}_0(Q)$. Then for every $\epsilon > 0$ there exists $\nu \in \mathcal{M}_0(Q)$ such that*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \epsilon \text{ and } \nu = \omega_t - \Delta\, \omega \text{ in } \mathcal{D}'(Q), \tag{8.1.5}$$

where $\omega \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$.

PROOF. See **[PPP2]**, Theorem 1.1. □

Note that we can apply Theorem 8.7 to construct a measurable function $u : Q \to \mathbb{R}$ such that the truncations $T_k^m(u)$ satisfy

$$(T_k^m(u))_t - \operatorname{div}(d(u)DT_k^m(u)) = (T_k^m(u))_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial T_k^m(u)}{\partial x_i}) = \mu + \Lambda_k + \Gamma \text{ in } Q, \tag{8.1.6}$$

for sequence of measures $\Lambda_k \in \mathcal{M}_b(Q)$ and a measure $\Gamma \in \mathcal{M}(Q)$ such that

$$\|\Lambda_k\|_{\mathcal{M}_b(Q)} \to 0 \text{ and } \int_Q \varphi d\Gamma = 0 \quad \forall \varphi \in C_0^1([0,T[), \tag{8.1.7}$$

such a formulation, no more based on the decomposition (8.1.4), can be extended to problem (8.0.1) straight forwardly and turns out to be suitable to tackle the problem with absorption term $h(u)$. Let us recall the following notations that will be used throughout this Chapter: for any $k > 0$ and any positive real number $m, \eta, \sigma > 0$, the functions T_k^m, $h_{k,\eta}$ and Z_σ are defined by

$$T_k^m(s) = \begin{cases} s & \text{if } -k \leq s \leq m \\ m & \text{if } s \geq m \\ \text{affine} & \text{otherwise,} \end{cases} \quad h_{k,\eta}(s) = \begin{cases} 0 & \text{if } s \geq -k \\ -1 & \text{if } s \leq -k-\eta \\ \text{affine} & \text{otherwise,} \end{cases} \quad Z_\sigma(s) = \begin{cases} 0 & \text{if } s \leq m-2\sigma \\ 1 & \text{if } s \geq m-\sigma \\ \text{affine} & \text{otherwise.} \end{cases} \tag{8.1.8}$$

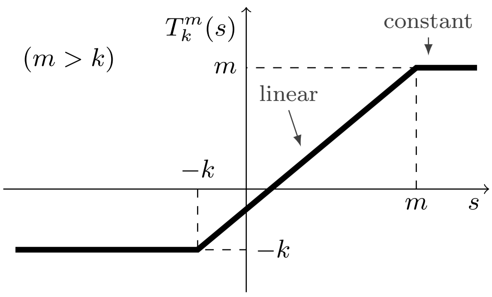


FIGURE 26. The function $T_k^m(s)$

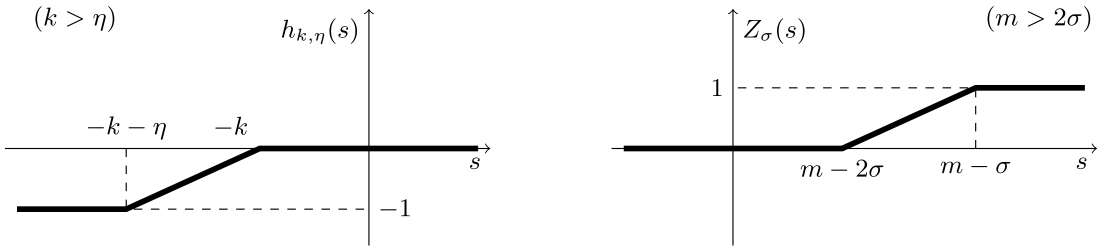


FIGURE 27. The functions $h_{k,\eta}(s)$ and $Z_\sigma(s)$

Finally, we will use the following notation for sequences $\omega(h, \eta, \delta, \cdots)$ to indicate any quantity that vanishes as the parameters go to their (obvious, if not explicitly stressed) limit point, with the same order in which they appear; that is, for instance

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \limsup_{h\to 0} |\omega(h, n, \delta)| = 0.$$

8.2. Main assumptions and renormalized formulation

Throughout this Chapter, we assume that Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $T > 0$ is a positive constant, $Q = \Omega \times (0, T)$ and $d(s) = (d_i(s))_{i=1}^N$ is a diagonal matrix defined on an interval $]-\infty, m[$ of $\mathbb{R}$ (m is a positive real number) with continuous coefficients $d_i(s)$ which satisfies the following assumptions

$$d_i \in C^0(]-\infty, m[;\ \mathbb{R}^+ \cup \{+\infty\}) \text{ with } d_i(s) < +\infty \quad \forall s < m\ \forall i \in \{1, \cdots, N\}, \tag{8.2.1}$$

$$\exists\ \alpha > 0 \text{ such that } d_i(s) \geq \alpha \quad \forall s \leq m\ \forall i \in \{1, \cdots, N\}, \tag{8.2.2}$$

$$\text{there exists an index } p \in \{1, \cdots, N\} \text{ such that } \lim_{s\to m^-} d_p(s) = +\infty \text{ and } \int_0^m d_p(s)ds < +\infty, \tag{8.2.3}$$

The initial data u_0 is defined on $L^1(\Omega)$ and is such that

$$u_0 \leq m \text{ a.e. in } \Omega, \tag{8.2.4}$$

and μ is a general measure, i.e.,

$$\mu \in \mathcal{M}_b(Q). \tag{8.2.5}$$

REMARK 8.8. The study of (8.0.1) under the assumption $\int_0^m d_p(s)ds = +\infty$ is easier (see **[VG3, Or]** and Remark 8.10) because one can then show that there exists a solution such that $u < m$ a.e. in Q. Assumption (8.2.3) imply that the $s-$dependent norm $|d^{\frac{1}{2}}(s)s|$ on $\mathbb{R}^N$ blows up as s tends to m uniformly.

Now, we give the definition of a renormalized solution of (8.0.1), this definition is more precise than the one used in **[ZR]** in the sense that it localizes the behaviour of the solution near the zone where the singular measure is concentrated.

DEFINITION 8.9. A measurable function u in $L^\infty(0, T; L^1(\Omega))$ is a renormalized solution of (8.0.1) if

$$T_k(u) \in L^2(0, T; H_0^1(\Omega)) \quad \forall k \geq 0, \tag{8.2.6}$$

$$u \leq m \text{ a.e. in } Q \quad \forall m \geq 0, \tag{8.2.7}$$

$$d(u)DT_k^m(u)\chi_{\{-k<u<m\}} \in (L^2(Q))^N \quad \forall k \geq 0, \tag{8.2.8}$$

there exists a sequence of nonnegative measures $(\Lambda_k) \in \mathcal{M}_b(Q)$ and a nonnegative measure $\Gamma \in \mathcal{M}(Q)$ such that

$$\lim_{k\to\infty} \|\Lambda_k\|_{\mathcal{M}_b(Q)} = \mu_s, \tag{8.2.9}$$

$$\int_Q \varphi d\Gamma = 0, \quad \forall \varphi \in C_0^1([0, T[), \tag{8.2.10}$$

and for every $k > 0$ and every $\varphi \in C_0^\infty([0,T) \times \Omega)$

$$(T_k^m(u))_t - \mathrm{div}(d(u)Du\chi_{\{-k<u<m\}}) = \mu_0 + \Lambda^k + \Gamma \tag{8.2.11}$$

in the sense of distributions.

REMARK 8.10. Note that

(i) Condition (8.2.6) is classical when dealing with renormalized solution for problems with measure data [**M, DPP, Pe1**]. The fact that $u \le m$ almost everywhere is Q is already explained and is natural using admissible test function $T_{2m}^+(u_n) - T_m^+(u_n)$ and the fact that $d_p(m - \frac{1}{n}) \underset{n\to\infty}{\to} +\infty$, which implies that $T_{2m}^+(u) - T_m^+(u) = 0$ a.e. in Q and then (8.2.7).

(ii) Condition (8.2.9) on the bahaviour of the energy near the set where μ_s is concentrated is an improvement of the one used in [**ZR**] when $\mu \in \mathcal{M}_0(Q)$.

(iii) Condition (8.2.11) means that $(T_k^m(u))_t - \mathrm{div}(d(u)Du\chi_{\{-k<u<m\}})$ is a bounded measure, then Λ^k is a diffuse measure. This is a key fact since it allows us to recover from (8.2.11) the standard estimates known for nonlinear potentials.

(iv) It is established in [**Pe1**] that since $u \in W_1$ then (the cap_2 quasi-continuous representative of) u is measurable with respect to μ, as a consequence, (8.2.11) makes sense and formally means that all terms have a meaning in $\mathcal{D}'(Q)$).

(v) The above analysis is restricted to the case where $\int_0^m d_p(s)ds < +\infty$.

Now, let us consider a solution u of (8.0.1) such that $u < m$ a.e. in Q and $u_0 < m$ a.e. in Ω. The usual technique to prove that u is a renormalized solution consists in plugging functions $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$ with $\varphi(T,x) = 0$. By Definition 8.9, we can use test function that depend on the solution itself in (8.2.11). Then reasoning as in Proposition 4.5 of [**PPP2**], renormalized solutions can be proved to be distributional solutions and enjoy the desired a priori estimates

PROPOSITION 8.11. *Let $\mu \in \mathcal{M}_0(Q)$, and $u_0 \in L^1(\Omega)$. Then the renormalized solution of problem (8.0.1) satisfies*

$$\begin{aligned}-\int_Q T_k^m(u)v_t\, dxdt - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u))\frac{\partial T_k^m(u)}{\partial x_i}\frac{\partial v}{\partial x_i}\chi_{\{-k<u<m\}}dxdt \\ = \int_Q \tilde{v}d\mu_0 + \int_Q \tilde{v}d\nu^k + \int_Q \tilde{v}d\Gamma + \int_\Omega T_k^m(u_0)v(0)dx.\end{aligned}$$

for every $\tilde{v} \in W \cap L^\infty(Q)$ such that $\tilde{v} = 0$ (with $\tilde{v}$ being the unique cap-quasi continuous representative of v).

PROOF. See [**PPP1**], Proposition 4.2. □

As a conclusion of this subsection, where $(d_i)_{i=1}^N$ are continuous, we claim that the notion of renormalized solutions and of weak solutions are equivalent. To prove this result we will use the following classical notion of renormalized solutions and weak solutions of (8.0.1).

DEFINITION 8.12. A measurable function u is a renormalized solution of (8.0.1) if, there exist a decomposition (f, G, g) of μ such that $v = u - g \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $q < \frac{N+2}{N+1}$,

$$v \le m \text{ a.e. in } Q \text{ and } T_k(v) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \text{ for every } k > 0, \tag{8.2.12}$$

$$d(u)Du\chi_{\{-k<u<m\}} \in (L^2(Q))^N, \tag{8.2.13}$$

and for every $S \in W^{2,\infty}(\mathbb{R})$ ($S(0) = 0$) such that S' has compact support on $\mathbb{R}$, we have

$$\begin{aligned}&\int_\Omega S(u_0)\varphi(0)dx - \int_0^T \langle \varphi_t, S(v)\rangle dt + \int_Q S'(v)d(u)Du \cdot D\varphi\chi_{\{u<m\}}dxdt \\ &+ \int_Q S''(v)d(u)Du \cdot Dv\varphi\, dxdt = \int_Q S'(v)\varphi d\mu_0\end{aligned} \tag{8.2.14}$$

for every $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$, with $\varphi(T,x) = 0$ such that $S'(v)\varphi \in L^2(0,T;H_0^1(\Omega))$. Moreover, for every $\psi \in C(\overline{Q})$ we have

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le v<2n\}} d(u)Du \cdot Dv\psi \, dxdt = \int_Q \psi d\mu_s^+, \\
&\lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n<v\le -n\}} d(u)Du \cdot Dv\psi \, dxdt = \int_Q \psi d\mu_s^-,
\end{aligned} \tag{8.2.15}$$

where μ_s^+ and μ_s^- are respectively the positive and the negative parts of the singular part μ_s of μ.

Remark 8.13. Note that

(i) Conditions (8.2.15) is the analog of (8.2.9).

(ii) Condition (8.2.14) is obtained through pointwise multiplication of (8.0.1) by $S'(v)$ (or, equivalently, by using $S'(v)\varphi$ as test function in (8.0.1) for any $\varphi \in C_c^\infty(Q)$), due to the properties of S' every term in (8.2.14) has a meaning in $L^1(Q) + L^2(0,T;H^{-1}(\Omega))$. Actually, we have $S'(v)d(u)Du\chi_{\{v<m\}} \in L^2(Q)^N$ because of (8.2.13), $S'(u)d(u)Du = S'(u)d(T_k^m(u))Du$ a.e. in Q for every $k > 0$.

(iii) Condition (8.2.14) may be equivalently replaced by (8.2.11) according to the interpretations of the various terms of (8.2.14).

(iv) Condition (8.2.15) prescribes the behaviour of μ_s near the sets where the parts μ_s^+ and μ_s^- (positive and negative parts of μ_s) are concentrated.

We recall the definition of a distributional solution of (8.0.1). Notice that such a definition makes sense for any measure μ, not necessarily diffuse, even if in our context we are always dealing with diffuse measures **[L]**.

Definition 8.14. If $\mu \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$, problem (8.0.1) has a unique solution in $W \cap C(0,T;L^2(\Omega))$ in the weak sense, that is

$$-\int_\Omega u_0\varphi(0)dx - \int_0^T \langle \varphi_t, u\rangle dt + \int_Q d(u)D(u)\cdot D\varphi \, dxdt = \int_0^T \langle \mu, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt,$$

for all $\varphi \in W$ such that $\varphi(T) = 0$

According to the similar arguments in **[PPP2]**, we have

Theorem 8.15. *A solution of* (8.0.1) *in the sense of Definitions 8.9 and 8.12 are equivalent and still equivalent to a weak solution of the same problem.*

Proof. To prove Theorem 8.15, it's easy to crossover the approach used in Theorem 4.11 in **[PPP2]**. □

8.3. Basic estimates and compactness results

In order to understand the meaning to give to the right hand side of (8.0.1), it is natural to look at what happens when we approximate the problem, that is when μ is replaced by a sequence μ_n of $C_c^\infty(Q)-$functions which converge to μ in the narrow topology (note that approximation in the weak–* topology of distributions would not be enough). We consider an approximation μ_n of μ which has the following properties: For every $(t,x) \in Q$ and $\mu \in \mathcal{M}_b(Q)$, we denote by $\rho_n * \mu$ the approximation of μ such that

$$\mu_n(t,x) = \rho_n * \mu(t,x) = \int_Q \rho_n(t-s, x-y)d\mu(s,y). \tag{8.3.1}$$

where (ρ_n) be a sequence of mollifiers satisfying

$$\rho_n \in C_c^\infty(\mathbb{R}^{N+1}),\ \text{Supp}\ \rho_n \subset B_{\frac{1}{n}}(0),\ \rho_n \ge 0 \text{ and } \int_{\mathbb{R}^{N+1}} \rho_n = 1. \tag{8.3.2}$$

Moreover, let introduce the following regularization: for $n \ge 1$ fixed

$$d_i^s(s) = d_i(T_{m-\frac{1}{n}}(s^+) - T_m(s^-)) \quad \forall s \in \mathbb{R}, \quad \forall i \in \{1,\cdots,N\}, \tag{8.3.3}$$

$$u_0^n \in C_c^\infty(\Omega): \quad u_0^n \to u_0 \text{ strongly in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty. \tag{8.3.4}$$

Let us call u_n the solution of problem

$$\begin{cases} (u_n)_t - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\right) = \mu_n & \text{in } (0,T)\times\Omega, \\ u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0,x) = u_0^n & \text{in } \Omega, \end{cases} \tag{8.3.5}$$

the existence of a solution of (8.3.5) can be readily studied by a straightforward application of Schauder's fixed point theorem. In fact, $u_n \in L^2(0,T;H_0^1(\Omega))$ verifies the variational formulation of problem (8.3.5) [**L**] which yields standard compactness results (see [**BDGO, DO2, DPP**]) that we collect in the following Proposition.

PROPOSITION 8.16. *Let u_n as defined before. Then*

$$\begin{cases} u_n \textit{ is bounded in } L^\infty(0,T;L^1(\Omega)), \\ T_k(u_n) \textit{ is bounded in } L^2(0,T;H_0^1(\Omega)), \\ d_i^n(u_n)^{\frac12}\dfrac{\partial T_k(u_n)}{\partial x_i} \textit{ is bounded in } L^2(Q), \\ d^n(u_n)DT_k(u_n) \textit{ is bounded in } (L^2(Q))^N. \end{cases} \tag{8.3.6}$$

Moreover, there exists a measurable function u such that $T_k(u) \in L^2(0,T;H_0^1(\Omega))$, u belong to $L^\infty(0,T;L^1(\Omega))$, and, up to a subsequence, for any $k>0$, we have

$$\begin{cases} u_n \to u \textit{ a.e. in } Q, \\ T_k(u_n) \rightharpoonup T_k(u) \textit{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ (d^n(u_n))^{\frac12}DT_k(u_n) \rightharpoonup d(u)^{\frac12}DT_k(u) \textit{ weakly in } (L^2(Q))^N, \\ d^n(u_n)DT_k^m(u_n) \rightharpoonup d(u)DT_k^m(u) \textit{ weakly in } (L^2(Q))^N. \end{cases} \tag{8.3.7}$$

Sketch of the proof. Here we give just an idea on how (8.3.6) can be obtained following the outlines of [**ZR**]. First of all, we choose $T_k(u_n)$ as test function in (8.3.5) to get

$$\int_\Omega \Theta_k(u_n)(t)dx + \sum_{i=1}^N \int_0^t\int_\Omega d_i^n(u_n)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^2 dxdt \le \int_0^t\int_\Omega \mu_n T_k(u_n)dxdt + \int_\Omega \Theta_n(u_0^n)dx \tag{8.3.8}$$

which yields from the fact that $\|u_0^n\|_{L^1(\Omega)}$ and $\|\mu_n\|_{L^1(Q)}$ are bounded

$$\int_\Omega \Theta_k(u_n)(t)dx + \sum_{i=1}^N \int_0^t\int_\Omega d_i^n(u_n)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^2 dxdt \le Ck. \tag{8.3.9}$$

Since $\Theta_k(s)\ge 0$ and $|\Theta_1(s)|\ge |s|-1$, we get

$$\int_\Omega |u_n(t)|dx + \sum_{i=1}^N \int_0^t\int_\Omega d_i^n(u_n)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^2 dxdt \le C(k+1) \quad \forall k>0 \quad \forall t\in[0,T]. \tag{8.3.10}$$

Taking the supremum in $[0,T]$, we obtain the estimate of u_n in $L^\infty(0,T;L^1(\Omega))$, of $T_k(u_n)$ in $L^2(0,T;H_0^1(\Omega))$ and of $d_i^n(u_n)^{\frac12}\frac{\partial T_k(u_n)}{\partial x_i}$ in $L^2(Q)$. Similarly we can get the estimate on $d^n(u_n)DT_k^m(u_n)$; let us choose $\int_0^{u_n} d_i^n(s)\chi_{\{-k\le s\le m\}}ds$ as test function in (8.3.5). Integrating on Q (recall that $\int_\Omega\int_0^{u_n}\int_0^z d_i^n(s)\ dsdzdx$ is positive and $\|\mu_n\|_{L^1(Q)}$ and $\|u_0^n\|_{L^1(\Omega)}$ are bounded) and using the fact that

$$|\int_0^{u_n} d_i^n(s)\chi_{\{-k\le s\le m\}}dx| \le \int_{-k}^m d_i(s)ds = C_k < +\infty$$

we have

$$\begin{aligned} &\int_\Omega\int_0^{u_n}\int_0^z d_i^n(s)\chi_{\{-k\le s\le m\}}dsdzdx + \int_Q (d_i^n(u_n))^2\left|\frac{\partial T_k^m(u_n)}{\partial x_i}\right|^2 dxdt \\ &\le \left(\|\mu_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right)\max_{i\in\{1,\dots,N\}}\int_{-k}^m d_i(s)ds \\ &\le C\max_{i\in\{1,\dots,N\}}\int_{-k}^m d_i(s)ds, \end{aligned}$$

which implies the estimate of $d^n(u_n)DT_k^m(u_n)$ in $(L^2(Q))^N$.

Remark 8.17. Let us observe that from above that, thanks to (8.3.6) and Stampacchia's theorem, we easily deduce that

$$\begin{cases} S(u_n) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \\ (S(u_n))_t \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{8.3.11}$$

Now our aim is to prove the following result.

Theorem 8.18. *Under the assumptions (8.2.1)-(8.2.5), there exists a renormalized solution of (8.0.1) in the sense of Definition 8.9.*

8.4. Proof of the main result

In this part we shall prove the existence of renormalized solutions of problem (8.0.1), to do that we will crossover the approach used in [**PPP2**] and [**ZR**] for diffuse measures with the one in [**Pe3**]. Let us introduce another auxiliary functions that we will often use in this section; this functions can be introduced in terms of T_k^m, $h_{k,\eta}$ and Z_σ and defined as follows,

(8.4.1)

$$S_{k,\eta}(s)=\begin{cases}1 & \text{if } s\geq -k\\ 0 & \text{if } s\leq -k-\eta\\ \text{affine} & \text{otherwise,}\end{cases}\quad S_{k,\sigma}^{m,\eta}(s)=\begin{cases}1 & \text{if } -k+\eta\leq s\leq m-2\sigma\\ 0 & \text{if } s\leq -k \text{ and } s\geq m-\sigma\\ \text{affine} & \text{otherwise,}\end{cases}\quad T_{k,\sigma}^{m,\eta}(s)=\int_0^z S_{k,\sigma}^{m,\eta}(s)ds.$$

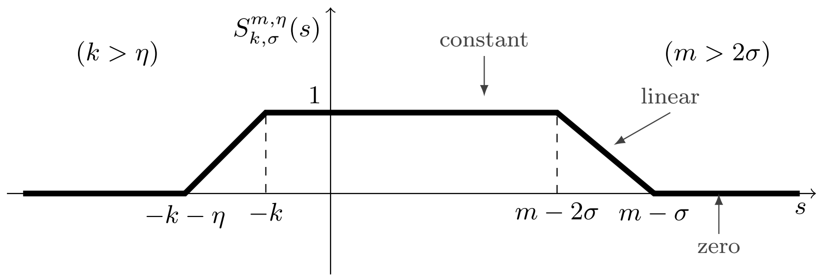


Figure 28. The function $S_{k,\sigma}^{m,\eta}(s)$

Then we have the following technical result whose proof can be obtained as in [**Pe1**].

Lemma 8.19. *Let μ_s be a nonnegative bounded Radon measure concentrated on a set of zero* $2-$*capacity. Then, for any $\delta>0$, there exists a compact set $K_\delta\subseteq E$ and a function $\psi_\delta\in C_c^\infty(Q)$ such that*

$$\mu_s(E\backslash K_\delta)\leq\delta,\quad 0\leq\psi_\delta\leq 1\quad \psi_\delta\equiv 1 \textit{ on } K_\delta,$$

and

$$\psi_\delta\to 0 \textit{ in } S \textit{ as } \delta\to 0$$

Moreover,

$$\int_Q (1-\psi_\delta)d\mu_s=\omega(s)$$

Proof. See [**Pe1**], Lemma 5. ☐

Proof of Theorem 8.18. The proof follow from [**DPP, PPP2, Pe1**] by a quite standard argument. We shall prove it in several steps.

Step 1. Estimates in $L^1(Q)$ on the energy term. For fixed $0<\eta<1$ and $0<\sigma<1$, we take $h_{k,\eta}(u_n)$ and $Z_\sigma(u_n)$ in (8.3.5) to obtain

$$\begin{aligned}\int_\Omega \overline{h}_{k,\eta}(u_n(T)) + \frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\le u_n\le -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \\ = \int_Q \mu_n h_{k,\eta}(u_n) + \int_\Omega \overline{h}_{k,\eta}(u_0^n)dx,\end{aligned} \tag{8.4.2}$$

and

$$\begin{aligned}\int_\Omega \overline{Z}_\sigma(u_n(T)) + \frac{1}{\sigma}\sum_{i=1}^N \int_{\{m-2\sigma\le u_n\le m-\sigma\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \\ = \int_Q \mu_n Z_\sigma(u_n)\mu_n dxdt + \int_\Omega \overline{Z}_\sigma(u_0^n)dx\end{aligned} \tag{8.4.3}$$

where $\overline{h}_{k,\eta}(s)=\int_0^s h_{k,\eta}(r)dr$ and $\overline{Z}_\sigma(s)=\int_0^s Z_\sigma(s)ds$ are respectively the primitives of the continuous functions $h_{k,\eta}(s)$ and $Z_\sigma(s)$. Observing that both terms in the left hand side of the above equalities are nonnegative, thanks to properties of $h_{k,\eta}$ and Z_σ, we have

$$\begin{cases}\displaystyle\frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\le u_n\le -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \le \int_{\{u_n\le -k\}}|\mu_n|dxdt + \int_{\{u_0^n\le -k\}}|u_0^n|dx,\\ \displaystyle\frac{1}{\sigma}\sum_{i=1}^N \int_{\{m-2\sigma\le u_n\le m-\sigma\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \le \int_{\{u_n\ge m-2\sigma\}} Z_\sigma(u_n)\mu_n dxdt + \int_{\{u_0^n\ge m-2\sigma\}}|u_0^n|dx,\end{cases} \tag{8.4.4}$$

while, since μ_n and u_0^n are bounded in $L^1(Q)$, we easily obtain

$$\begin{cases}\displaystyle\frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\le u_n\le -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \le C_1,\\ \displaystyle\frac{1}{\sigma}\sum_{i=1}^N \int_{\{m-2\sigma\le u_n\le m-\sigma\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 dxdt \le C_2.\end{cases} \tag{8.4.5}$$

Thus, there exists a bounded Radon measures λ_n^k and ν_σ such that, as η ans σ tends to zero

$$\begin{cases}\displaystyle\frac{1}{\eta}\sum_{i=1}^N d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 \chi_{\{-k-\eta\le u_n\le -k\}} \rightharpoonup \lambda_n^k \quad \text{weakly--* in } \mathcal{M}(Q),\\ \displaystyle\frac{1}{\sigma}\sum_{i=1}^N d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 \chi_{\{m-2\sigma\le u_n\le m-\sigma\}} \rightharpoonup \nu_\sigma \quad \text{weakly--* in } \mathcal{M}(Q).\end{cases} \tag{8.4.6}$$

Step 2. Equations for the truncations. Now we want to check that (8.2.11) holds true for u. For all real numbers $\eta>0$, $\sigma>0$ and $k>0$, we multiply (8.3.5) by $S_{k,\sigma}^{m,\eta}(u_n)\varphi$, where $\varphi\in C_c^\infty([0,T]\times\Omega)$, to obtain, after passing to the limit as η tends to zero

$$\begin{aligned}(T_{k,\sigma}^m(u_n))_t - \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(d_i^n(u_n)\frac{\partial T_{k,\sigma}^m(u_n)}{\partial x_i}\Big) - \mu_0^n - \mu_0^n Z_\sigma(u^n) \\ = \lambda_k^n + \mu_s^n\chi_{\{u_n>-k\}} - \mu_0^n\chi_{\{u_n\le -k\}} \\ + \nu_\sigma^n + \mu_s^n Z_\sigma(u^n)\chi_{\{u_n<m-2\sigma\}} - \mu_0^n Z_\sigma(u^n)\chi_{\{u_n\ge m-2\sigma\}}\end{aligned} \tag{8.4.7}$$

in $\mathcal{D}'(Q)$. We define the measures Λ_k^n and Γ_σ^n as

$$\begin{cases}\Lambda_k^n := \lambda_k^n + \mu_s^n\chi_{\{u_n>-k\}} - \mu_0^n\chi_{\{u_n\le -k\}}\\ \Gamma_\sigma^n := \nu_\sigma^n + \mu_s^n Z_\sigma(u^n)\chi_{\{u_n<m-2\sigma\}} - \mu_0^n Z_\sigma(u^n)\chi_{\{u_n\ge m-2\sigma\}}\end{cases} \tag{8.4.8}$$

Notice that

$$\|\Lambda_k^n\|_{L^1(Q)} \le C, \quad \|\Gamma_\sigma^n\|_{L^1(Q)} \le C. \tag{8.4.9}$$

So that there exists Λ_k and Γ_σ in $\mathcal{M}(Q)$ such that

$$\begin{cases} \Lambda_k^n \rightharpoonup \Lambda_k & \text{weakly–* in } \mathcal{M}(Q), \\ \Gamma_\sigma^n \rightharpoonup \Gamma_\sigma & \text{weakly–* in } \mathcal{M}(Q). \end{cases} \tag{8.4.10}$$

Therefore, from (8.3.7) we deduce that

$$(T_{k,\sigma}^m(u))_t - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial T_{k,\sigma}^m(u)}{\partial x_i}\chi_{\{-k<u<m\}}) = \mu_0 + \Lambda_k + \Gamma_\sigma \text{ in } \mathcal{D}'(Q) \tag{8.4.11}$$

Note that

$$\begin{aligned} \int_Q |\Gamma_\sigma| dxdt &\le \liminf_{n\to+\infty} \int_Q |\Gamma_\sigma^n| dxdt \\ &= \liminf_{n\to+\infty} \int_Q |\nu_\sigma^n - \mu_n Z_\sigma(u_n)| dxdt \\ &\le 2\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(\Omega)} \end{aligned} \tag{8.4.12}$$

Then there exists a bounded measure Γ such that

$$\Gamma_\sigma \rightharpoonup \Gamma \quad \text{weakly–* in } \mathcal{M}(Q)$$

Therefore, after taking the limit as σ vanishes

$$(T_k^m(u))_t - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial T_k^m(u)}{\partial x_i}\chi_{\{-k<u<m\}}) = \mu_0 + \Lambda_k + \Gamma \text{ in } \mathcal{D}'(Q) \tag{8.4.13}$$

Step 3. The limit of Λ_k *and* Γ. Let us consider the distributional formulation of (8.3.5) and let us subtract (8.4.13) from it, to obtain, for any $\varphi \in C_c^\infty([0,T]\times\Omega)$

$$\begin{aligned} &-\int_Q (u_n - T_k^m(u))\varphi_t dxdt + \int_Q \sum_{i=1}^N (d_i(u_n)\frac{\partial u_n}{\partial x_i} - d_i(u)\frac{\partial T_k^m}{\partial x_i}\chi_{\{-k<u<m\}})\frac{\partial \varphi}{\partial x_i} dxdt \\ &= \int_Q \varphi d(\mu_0^n - \mu_0) + \int_Q \varphi d(\mu_s^n - \Lambda_k) - \int_Q \varphi d\Gamma + \int_\Omega \varphi(0)(u_0^n - T_k^m(u_0))dx. \end{aligned}$$

For any function $\varphi \in C_0^1([0,T[)$, we have

$$\int_Q \varphi d\Gamma = \int_Q \varphi d\Gamma_\sigma + \omega(\sigma) = \int_Q \varphi d\Gamma_\sigma^n dxdt + \omega(\sigma, n) \tag{8.4.14}$$

where $\Gamma_\sigma^n = \frac{1}{\sigma}\sum_{i=1}^N d_i^n(u_n)|\frac{\partial u_n}{\partial x_i}|^2 \chi_{\{m-2\sigma<u_n<m-\sigma\}} - Z_\sigma(u_n)\mu_n$.

Remark that since $Z_\sigma(u_n)\varphi$ is an admissible test function in (8.4.5), since $\varphi \in C_0^1([0,T[)$

$$\int_\Omega \overline{Z}_\sigma(u_0^n)\varphi(0)dx + \int_Q \overline{Z}_\sigma(u_n)\varphi_t dxdt = \int_Q \varphi \Gamma_\sigma^n dxdt, \tag{8.4.15}$$

due to the fact that

$$\begin{aligned} \overline{Z}_\sigma(u_n) &\to \overline{Z}_\sigma(u) \text{ in } L^1(Q) \text{ as } n \to \infty, \\ \overline{Z}_\sigma(u_0^n) &\to \overline{Z}_\sigma(u_0) \text{ in } L^1(\Omega) \text{ as } n \to \infty, \end{aligned}$$

then $\int_Q \overline{Z}_\sigma(u_n)\varphi_t dx$ converges to $\int_Q \overline{Z}_\sigma(u)\varphi_t dx$ and $\int_\Omega \overline{Z}_\sigma(u_0^n)\varphi dx$ to $\int_\Omega \overline{Z}_\sigma(u_0)\varphi dx$ as n tends to infinity. Since $\overline{Z}_\sigma(u)$ converges to $(u-m)^+$ and $u \le m$, $u_0 \le m$ a.e, then

$$\begin{cases} \displaystyle\int_Q \overline{Z}_\sigma(u_n)\varphi_t dx = \int_Q (u-m)^+\varphi_t dx = \omega(\sigma, n), \\ \displaystyle\int_Q \overline{Z}_\sigma(u_0^n)\varphi dx = \int_\Omega (u_0-m)^+\varphi dx = \omega(\sigma, n). \end{cases} \tag{8.4.16}$$

Then, from (8.4.14), (8.4.15) and (8.4.16) we have

$$\int_Q \varphi d\Gamma = 0 \quad \forall \varphi \in ([0,T[).$$

Using (8.3.7), we are able to pass to the limit in the above equality as n tends to $+\infty$ and to establish

$$\left\{\begin{aligned}&\int_Q \varphi d\Lambda_k = \int_Q \varphi d\mu_s + \omega(n,k) \quad \text{ for all } \varphi \in C_0^\infty(Q),\\ &\int_Q \varphi d\Gamma = 0 \quad \text{ for all } \varphi \in C_0^1([0,T[).\end{aligned}\right. \tag{8.4.17}$$

Finally, we have to prove that the previous limit is true in measure. Let us choose without loss of generality $\varphi \in C^1(\overline{Q})$, reasoning by density, for every $\varphi \in C(\overline{Q})$, and using cut-off functions ψ_δ defined in Lemma 8.19

$$\int_Q \varphi d\Lambda_k = \int_Q \varphi\psi_\delta d\Lambda_k + \int_Q \varphi(1-\psi_\delta) d\Lambda_k, \tag{8.4.18}$$

then

$$\int_Q \varphi\psi_\delta d\Lambda_k = \int_Q \varphi\psi_\delta d\mu_s + \omega(k). \tag{8.4.19}$$

While, by construction of ψ_δ (i.e. $\psi_\delta = 1$ on K_δ), we have

$$\int_Q \varphi\psi_\delta d\mu_s = \int_{K_\delta} \varphi d\mu_s + \int_{E\setminus K_\delta} \varphi\psi_\delta d\mu_s.$$

On the other hand, Proposition 8.16 and the Lebesgue convergence Theorem implies

$$\int_{E\setminus K_\delta} \varphi\psi_\delta d\mu_s \leq \delta\|\varphi\|_{L^\infty(Q)} \text{ and } \int_{K_\delta} \varphi d\mu_s = \int_Q \varphi d\mu_s = \omega(\delta).$$

Putting together last results with (8.4.14), we get

$$\int_Q \varphi\psi_\delta d\Lambda_k = \int_Q d\mu_s + \omega(k,\delta).$$

Step 4. Proof completed. Let us now prove

$$\int_Q \varphi(1-\psi_\delta) d\Lambda_k = \omega(k,\delta). \tag{8.4.20}$$

Using the definition of Λ_k, we see that

$$\begin{aligned}\int_Q \varphi(1-\psi_\delta) d\Lambda_k = \lim_{n\to\infty} [\lim_{\eta\to 0} \frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\leq u_n\leq -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 \varphi(1-\psi_\delta))dxdt\\ + \int_{\{u_n\leq -k\}} \varphi(1-\psi_\delta) d\mu_s^n - \int_{\{u_n > -k\}} \varphi(1-\psi_\delta) d\mu_0^n].\end{aligned} \tag{8.4.21}$$

As a consequence of Lemma 8.4 and the fact that μ_0^n are equi-diffuse measures, we obtain

$$\int_{\{u_n>-k\}} \varphi(1-\psi_\delta) d\mu_0^n = \omega(n,k). \tag{8.4.22}$$

Finally, we have to prove that

$$\frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\leq u_n\leq -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 \varphi(1-\psi_\delta) dxdt = \omega(\eta,n,k,\delta). \tag{8.4.23}$$

To do that, we use again (8.3.5) with test functions $h_{k,\eta}(u_n)(1-\psi_\delta)$, we have
(8.4.24)

$$\begin{aligned}\int_Q \overline{h}_{k,\eta}(u_n(t,x))(\psi_\delta)_t dxdt - \int_\Omega \overline{h}_{k,\eta}(u_0^n)(1-\psi_\delta(0)) + \frac{1}{\eta}\sum_{i=1}^N \int_{\{-k-\eta\leq u_n\leq -k\}} d_i^n(u_n)\left|\frac{\partial u_n}{\partial x_i}\right|^2 (1-\psi_\delta) dxdt\\ - \sum_{i=1}^N \int_Q d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\delta}{\partial x_i} h_{k,\eta}(u_n) dxdt = \int_Q \mu_0^n h_{k,\eta}(u_n)(1-\psi_\delta) dxdt + \int_Q \mu_s^n h_{k,\eta}(u_n)(1-\psi_\delta) dxdt,\end{aligned}$$

we see that u_n and $|d_i^n(u_n)\frac{\partial u_n}{\partial x_i}|$ converges in $L^1(Q)$. The properties of ψ_δ allow then to conclude that

$$\begin{cases} \displaystyle\int_Q \overline{h}_{k,\eta}(u_n(t,x))(\psi_\delta)_t dxdt = \omega(n,k), \\ \displaystyle\sum_{i=1}^N \int_Q d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\delta}{\partial x_i} h_{k,\eta}(u_n) dxdt = \omega(n,k). \end{cases}$$

Similarly, let us remark that at $t = 0$

$$\int_\Omega \overline{h}_{k,\eta}(u_0^n(x))(1-\psi_\delta(0))dx = \omega(n,k).$$

Now, thanks to Lemma 8.4 and properties of the equi-diffuse measure μ_0^n, we readily have

$$\begin{cases} \displaystyle\left|\int_Q \mu_0^n h_{k,\eta}(1-\psi_\delta)\right| \le \int_{\{u_n\le -k\}} \mu_0^s(1-\psi_\delta) = \omega(n,k), \\ \displaystyle\left|\int_Q \mu_s^n h_{k,\eta}(u_n)(1-\psi_\delta)\right| \le \int_Q \mu_s^n(1-\psi_\delta) = \omega(n,\delta), \end{cases}$$

Collecting together all these results to obtain (8.4.23).

APPENDIX A

Remarks, conclusion and perspectives

1. Uniqueness of renormalized solutions

Most uniqueness results are available in literature when $\mu \in \mathcal{M}_0(\Omega)$, the uniqueness of renormalized solutions has been proved in [**BGO1, LM, M**]. However these are only few references on a vast literature on the subject. Note that the uniqueness is a hardest task in the framework of renormalized solutions and general measure data. It follows from [**DMOP**] that if a satisfies further hypothesis, namely the strong monotonicity and the local Lipschitz continuity, or the Hölder continuity with respect to ζ (these hypotheses are satisfied for example by the function $a(x,\zeta) = |\zeta|^{p-2}\zeta$) and if $u - \tilde{u} \in L^\infty(\Omega)$ (the precise meaning of the fact that two solutions are comparable), then $u = \tilde{u}$. This condition can be localized in a neighborhood $\mathcal{U}$ of the set where the singular measure μ is concentrated and that it is sufficient to assume that $(u-\tilde{u})^-$ (the negative part of $u - \tilde{u}$) belongs to $L^\infty(\mathcal{U})$, see [**O**]. Note also that, in the proof of such results, the test functions $T_k(u-\tilde{u}) \in W_0^{1,p}(\Omega)$ are needed to ensure uniqueness. Moreover the results of [**AA3, AA5**] overlap with the one obtained in [**Pe1**] for parabolic case are obtained without uniqueness. In order to perform the uniqueness, there are several technical difficulties in the proof of the equivalence results of renormalized solutions as stated in the elliptic case [**DMOP**].

2. Diffuse measure and nonlinear parabolic problems with variable exponent

The representation result proved in [**OT**] states the following: if μ is a diffuse measure, then there exist $f \in L^1(Q)$, $F \in (L^{p'(\cdot)}(Q))^N$, $g \in L^{(p_-)'}(0,T;V)$ and $\chi \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$ such that

$$\int_Q \varphi d\mu = \int_Q f\varphi \, dxdt + \int_Q F\cdot\nabla\varphi \, dxdt + \int_0^T \langle \chi,\varphi\rangle dt - \int_0^T \langle \varphi_t, g\rangle \quad \forall \varphi \in C_c^\infty([0,T]\times\Omega). \tag{A.2.1}$$

The possibility that the above decomposition holds for some $g \in L^\infty(Q)$, has a special interest, as it was also pointed out in Chapter 3. In particular, one has the following counterpart

Proposition A.1. *Assume that $\mu \in \mathcal{M}_0(Q)$ satisfies (A.2.1), where $f \in L^1(Q)$, $g \in L^{p_-}(0,T;V)$ and $\chi \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$. If $g \in L^\infty(Q)$, then μ is diffuse.*

Let us illustrate the main situation that can be treated in the spirit of this Chapter. In that case, we can prove that the solution $u(t,x)$ exists for all positive times $t > 0$ for the parabolic problem with absorption term and exponent variable. More precisely, the following model problem

$$\begin{cases} u_t - \Delta_{p(\cdot)}u + h(u) = \mu & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{A.2.2}$$

where Ω be an open bounded subset of $\mathbb{R}^N$, $T>0$, $p(\cdot) : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $1 < p_- \leq p_+ < +\infty$, where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$, $\Delta_{p(\cdot)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(\cdot)-$Laplace operator and μ is a bounded Radon measure in $Q = (0,T)\times\Omega$, $u_0 \in L^1(\Omega)$, and $h : \mathbb{R}\to\mathbb{R}$ is a continuous function such that $h(s)s \geq 0$ for large $|s|$ using the capacitary estimate of Lemma 3.9 and the same arguments as [**PPP2**].

3. Renormalized solutions for parabolic problems with general form of measures

A possible extension of the result of Chapter 4 could be the proof of existence of a renormalized solution for problem

$$\text{(A.3.1)}\qquad \begin{cases} u_t - \operatorname{div}(a(t,x,u,\nabla u)) = H(u)\mu & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \end{cases}$$

where H is a continuous, positive bounded function (i.e. $H \in C_b^0(\mathbb{R})$) and μ is a general Radon measure. The operator $u \mapsto -\operatorname{div}(t,x,u,\nabla u)$ is a monotone, coercive and with growth in u and its gradient ∇u, motivated by control problems arising in chemical reactions [**MT**], the authors in [**MPo**] prove under the assumption that H has a limit at infinity

$$\text{(A.3.2)}\qquad H \in C_b^0(\mathbb{R}), \quad H(s) > 0\ \forall s \in \mathbb{R}, \quad \exists \lim_{s\to+\infty} H(s) = H(\infty)$$

the convergence of approximate solutions towards a function u which solves the equation

$$\text{(A.3.3)}\qquad \begin{cases} -\operatorname{div}(a(x,\nabla u)) = H(u)\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ belong to $\mathcal{M}_b^+(\Omega)$ splitted as $\mu = \mu_0 + \lambda = f - \operatorname{div}(F) + \lambda$, with $\lambda \geq 0$ concentrated on E with $\operatorname{cap}_p(E) = 0$. Looking for the asymptotic behaviour, as ϵ tends to zero, of the approximating Dirichlet problem

$$\text{(A.3.4)}\qquad \begin{cases} -\operatorname{div}(a(x,\nabla u_\epsilon)) = H(u_\epsilon)\mu_\epsilon & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_ϵ is a reasonable smooth approximation of μ, like for instance a standard convolution of μ with a mollifying Kernel, a compactness result of the sequence of solutions of (A.3.4) is obtained, that is there exists a subsequence u_ϵ of solutions of (A.3.4) converging to a function u such that $H(u)$ is μ_0-measurable (and hence belong to $L^\infty(\Omega, d\mu_0)$) and such that, if $H(\infty) > 0$, the function u blows upon the set where λ is concentrated. This suggest that the product $H(u)\mu$ should be formally written as $H(u)\mu = H(u)\mu_0 + H(\infty)\lambda$. However, it should be observed that a straightforward consequence is that, when $H(\infty) = 0$, a same function u is a solution of equation (A.3.3) relative to all measures μ having the same regular part μ_0 but possibly different singular part λ. In other terms, if $H(\infty) = 0$, the singular parts λ "disappear", as ϵ tends to zero, in the limit problem of equation (A.3.3); for instance, if μ is a Dirac mass, the solutions u_ϵ of equation (A.3.4) converge to zero. We recall the following result proved by the approach based on the stability properties of [**MPo**].

Theorem A.2. *Assume that (A.3.2) hold true. Then there exists $u \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$, $H(u)$ belong to $L^\infty(\Omega, d\mu_0)$, and for a subsequence u_ϵ solutions of (A.3.4), we have*

$$\begin{cases} T_k(u_\epsilon) \to T_k(u) \text{ strongly in } H_0^1(\Omega) \text{ for every } k > 0, \\ u_\epsilon \to u \text{ strongly in } W_0^{1,q}(\Omega) \text{ for every } q < \dfrac{N}{N-1}, \\ \displaystyle\lim_{\epsilon\to 0} \int_\Omega \varphi H(u_\epsilon)\, d\mu_0^\epsilon = \int_\Omega \varphi H(u)\, d\mu_0 \text{ for every } \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\lim_{\epsilon\to 0} \int_\Omega \varphi H(u_\epsilon)\, d\lambda_\epsilon dx = H(\infty) \int_\Omega \varphi\, d\lambda \text{ for every } \varphi \in C_b^0(\Omega). \end{cases}$$

Moreover, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n<u<2n\}} a(x,\nabla u)\cdot\nabla u = H(\infty) \int_\Omega \varphi\, d\lambda, \quad \text{for every } \varphi \in C_b^0(\Omega).$$

Theorem A.2 suggests a setting for the definition of a solution of problem (A.3.3) [**MPo**], this setting is the natural extension of the framework of so-called renormalized solutions for measure data defined in [**DMOP**].

4. Standard porous problems with natural growth term

Similar results to those of Chapter 5 can be obtained for the initial boundary value problem

$$\begin{cases} b(u)_t - \operatorname{div}(a(t,x,u,\nabla u)) + g(u)|\nabla u|^p = \mu & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \end{cases} \tag{A.4.1}$$

where both μ and u_0 are, possibly singular, general measure data, b is a strictly increasing C^1-function and $-\operatorname{div}(a(t,x,u,\nabla u))$ is a Leray-Lions operator with growth $|\nabla u|^{p-1}$ in ∇u but without any growth assumption on u and the function g is just assumed to be continuous on $\mathbb{R}$ and to satisfy a sign condition. This should be done using a method of Chapter 4 with the one of **[AR]** where the authors prove the existence of renormalized solutions for problem (A.4.1) where $\mu \in L^1(Q)$ and $b(u_0) \in L^1(\Omega)$ and **[BP]** where $b(u) = u$, $\mu \in L^1(Q)$ and $b(u_0) = u_0$ is a general measure in $\mathcal{M}_b(Q)$.

5. Generalized fractional porous medium problems

The interest of fractional Sobolev spaces has constantly increased over the last years. These spaces arise in a number of applications such as phase transition, quasi-geostrophic flows and quantum mechanics, see **[Sil]** and references therein for more applications. Recently, motivated by some new Laplacian operators, called "fractional $p(x)$-Laplacian" arising in continuum mechanics introduced in **[KRV]**, and used to obtain existence and uniqueness of nonnegative (renormalized) solutions for elliptic equations with integrable data $f \in L^1(\Omega)$ in **[ZZ1]**. Formally, the fractional $p(x)-$Laplacian of order s of a function $u \in W^{s,p(x,y)}(\Omega)$ is defined as

$$(-\Delta)^s_{p(\cdot)}u(x) = \mathrm{P.V.}\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy,\ x\in\Omega,$$

where P.V. is the principal value, s is a fixed real number such that $0 < s < 1$, $p(\cdot) : \overline{\Omega}\times\overline{\Omega} \to]1,+\infty[$ is a continuous function with $sp(x,y) < N$ for any $(x,y) \in \overline{\Omega}\times\overline{\Omega}$. We define a weak solution $u \in W^{s,p(x,y)}(\Omega)$ of the problem $(-\Delta)^s_{p(\cdot)}u = f$ if there exist $u \in W_0^{s,p(x,y)}(\Omega)$ such that

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy = \int_\Omega f\varphi dx$$

for all $\varphi \in W_0^{s,p(x,y)}(\Omega)$. Now consider the evolution case with Dirichlet boundary conditions in $Q = (0,T)\times\Omega$, where Ω a bounded domain, we have

Theorem A.3. *Assume that $1 < p_- \le p_+ < \infty$. For every $u_0 \in L^2(\Omega)$, there exists a unique renormalized solution of the parabolic problem*

$$\begin{cases} u_t(t,x) + (-\Delta)^s_{p(\cdot)}u(t,x) = f & \textit{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \textit{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \textit{in } \Omega, \end{cases} \tag{A.5.1}$$

Similar existence results (we refer to Chapter 6 for the precise statements) can be obtained for general boundary value problems, that is, when we consider

$$\begin{cases} b(x,u)_t - \mathcal{L}_{p(\cdot)}u(t,x) = \mu & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \end{cases}$$

where $b(x,u)$ is a general unbounded term depending on t, x and u with double derivatives $\nabla_x b : \Omega\times\mathbb{R} \to \mathbb{R}^N$ and $b_s : \Omega\times\mathbb{R}\to\mathbb{R}$. In this case the asymptotic behaviour is given by using the non-local operator

$$-\mathcal{L}_{p(\cdot)}u(t,x) = \mathrm{P.V.}\int_Q |u(x,t)-u(y,t)|^{p(x,y)-2}(u(x,t)-u(y,t))k(x,y)dy,$$

where the kernel $k : \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ is assumed to be measurable, and satisfies the following coercivity condition

$$\frac{1}{\Lambda|x-y|^{N+sp(x,y)}} \le k(x,y) \le \frac{\Lambda}{|x-y|^{N+sp(x,y)}} \quad \forall x,y\in\Omega,\ x\ne y,\ \Lambda\ge 1.$$

6. Orlicz capacities for parabolic problems with absorption term

The foundations for the study of Orlicz capacity were established by Aïssaoui in [**Ais1**], Aïssaoui and Benkirane in [**AisB1, AisB2**] in view of the several applications in nonlinear potential theory, in harmonic analysis and also in PDE's theory. Notwithstanding, the subject is still of interest in recent years, see [**Ais2, MO**] for a broad treatment of the topic. In order to study the properties of the capacities, the comparison theorems turn out to be relevant (see Section 5.1 in [**AH**]). Moreover, this kind of results are a key tool in questions related to the existence of solutions for some nonlinear elliptic and parabolic problems involving measures [**FP, OP**]. The aim of this part is to give some contributions in these directions. We remark that we can extend the result of Chapter 7 to problems with lower order terms more general than $|u|^{q-1}u$ in the context of Orlicz spaces. The best approach for this new context will involve also the notion of Orlicz capacity. Such a notion has been already introduced in literature in [**AisB1**]. In spite of this, we can adopt a new equivalent definition of entropy solutions which is closer to the one used in classical Sobolev spaces. Denote by Q the parabolic cylinder $(0,T)\times\Omega$ with Ω is an open subset of $\mathbb{R}^N$, $0<\alpha<N$ and r be a real number with $r>1$. The $(\alpha,r)-$capacity of a compact subset K with respect to Q is defined [**AH**] as

$$\operatorname{cap}_{\alpha,r}(K)=\operatorname{cap}_{\alpha,r}(K,Q)=\inf\{\|u\|^r_{W_0^{\alpha,r}(\Omega}:\ u\in C_c^\infty(Q),\ u\geq\chi_K\},$$

where χ_K is the characteristic function of K, we will use the convention that $\inf\emptyset=+\infty$. The $(\alpha,r)-$capacity of any open subset U of Q is then defined by

$$\operatorname{cap}_{\alpha,r}(U)=\operatorname{cap}_{\alpha,r}(U,Q)=\sup\{\operatorname{cap}_{\alpha,r}(K),\ K\text{ compact},\ K\subset U\},$$

and the $(\alpha,r)-$capacity of any set $E\subset Q$ by

$$\operatorname{cap}_{\alpha,r}(E)=\operatorname{cap}_{\alpha,r}(E,Q)=\inf\{\operatorname{cap}_{\alpha,r}(U),\ U\text{ open},\ E\subset U\}.$$

The previous definitions can be generalized in the context of Orlicz spaces by using $N-$functions A.

Definition A.4. Let K be a compact subset of Q and let A be a $N-$function satisfying (1.18.1). The $(1,A)-$capacity of K with respect to Q is defined as

$$\operatorname{cap}_{1,A}(K)=\inf\{A(\|\nabla u\|_A):\ u\in C_c^\infty(Q),\ u\geq\chi_K\},$$

where χ_K is the characteristic function of K, we will use the convention that $\inf\emptyset=+\infty$. The $(1,A)-$capacity of any open subset U of Q is defined by

$$\operatorname{cap}_{1,A}(U)=\sup\{\operatorname{cap}_{1,A}(K),\ K\text{ compact},\ K\subset U\},$$

and the $(1,A)-$capacity of any set $E\subset Q$ by

$$\operatorname{cap}_{1,A}(B)=\inf\{\operatorname{cap}_{1,A}(U),\ U\text{ open},\ E\subset U\}.$$

Note that this formulations are equivalent (see [**Ais4, Ais5, AisB1, FP**]). Consider now a class of nonlinear parabolic problems more general than (7.0.1)

$$\text{(A.6.1)}\qquad\begin{cases}u_t-\operatorname{div}(a(t,x,\nabla u))+\Phi''(|u|)u=\mu & \text{in }(0,T)\times\Omega,\\ u(t,x)=0 & \text{on }(0,T)\times\partial\Omega,\\ u(0,x)=0 & \text{in }\Omega,\end{cases}$$

where $a:(0,T)\times\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot,\cdot,\zeta)$ is measurable on Q for every ζ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q), such that the following assumptions holds for some $N-$function A

$$\text{(A.6.2)}\qquad a(t,x,\zeta)\cdot\zeta\geq\alpha A(|\zeta|),$$

$$\text{(A.6.3)}\qquad |a(t,x,\zeta)|\leq c(t,x)+k_1\overline{A}^{-1}A(k_2|\zeta|),$$

$$\text{(A.6.4)}\qquad [a(t,x,\zeta)-a(t,x,\eta)]\cdot(\zeta-\eta)>0,$$

for almost every (t,x) in Q, for every ζ,η in $\mathbb{R}^N$ with $\zeta\neq\eta$, where α is a positive constant, $k_i\in\mathbb{R}^+$, for $i=1,2$ and $c(t,x)$ is a nonnegative function in $E_{\overline{M}(Q)}$ with the $N-$function $\overline{A}$ is the conjugate of A, notice that the problem (7.0.1) is obtained by taking $A(t)=t^p$. Define the differential operator $A(u)=-\operatorname{div}(a(t,x,\nabla u))$, under assumptions (A.6.2), (A.6.3) and (A.6.4), $u \mapsto -\mathrm{div}(a(t,x,\nabla u))$ is a uniformly parabolic, coercive, and pseudo-monotone operator acting from $W^{1,x}L_A(Q)$ to is dual $W^{-1,x}L_{\overline{A}(Q)}$ **[Ad]**, and so it is surjective **[LL]**. Note that μ is a bounded Radon measure concentrated on a set E of null $1, A-$capacity (i.e $\mu(B) = \mu(B \cap E)$ for every Borelian subset B of Q), note also that problem (A.6.1) with μ replaced by δ is obtained by taking $A(t) = t^N$ because δ is concentrated on a point whose $(1, N)-$capacity is zero. The definition of an entropy solution for problem (A.6.1) can be stated as follows

Definition A.5. Let $g \in L^1(Q)$ and $\lambda = 0$ and let $\Phi \in C^2([0,\infty[)$. A measurable function $u : Q \to \mathbb{R}$ is called entropy solution of (A.6.1) if u belongs to $L^\infty(0,T;L^1(\Omega))$, $T_k(u)$ belongs to $D(A) \cap W_0^{1,x}L_M(Q)$ for every $k > 0$, $\Theta_k(u)$ belongs to $L^1(\Omega)$ for every $t \in [0,T[$. Moreover $\Phi''(|u|)|u|$ belongs to $L^1(Q)$ and for every $k > 0$

$$\begin{aligned}&\int_\Omega \Theta_k(u-\varphi)(t,x)dx - \int_\Omega \Theta_k(u-\varphi)(0,x)dx \\ &+ \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle dt + \int_Q a(t,x,\nabla u)\cdot \nabla T_k(u-\varphi)dxdt \\ &+ \int_Q \Phi''(|u|)uT_k(u-\varphi)dxdt \le \int_Q gT_k(u-\varphi)dxdt,\end{aligned}$$

and the initial condition satisfies

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \Omega,$$

for every $\varphi \in W_0^{1,x}L_A(Q) \cap L^\infty(Q)$ such that φ_t belongs to $W^{-1,x}L_{\overline{A}}(Q) + L^1(Q)$ (recall that $\Theta_k(r) = \int_0^r T_k(s)ds$ is the primitive of the usual truncation T_k).

The question now is the following: Let u satisfy the assumptions above, A be an $N-$function, and λ be a bounded measure concentrated on a set E of null $A-$capacity. Let f_n be a sequence of functions converging to λ in the sense of (7.1.7), g be a function in $L^1(Q)$ and g_n be a sequence of $L^\infty(Q)-$functions which converge to g weakly in $L^1(Q)$. What happens if $\Phi \in C^2([0,\infty[)$ is a $N-$function ? What are the conditions on Φ for the analog result of Theorem 7.2 ? For the proof of the nonexistence result, we can construct, as in Lemma 7.10, a sequence of suitable cut-off functions with the conditions $\|\nabla \psi_\delta^+\|_A \le \delta$ and $\|\nabla \psi_\delta^-\|_A \le \delta$.

7. Diffusion parabolic problems with singular coefficients

In this part we propose some possible extensions of Chapter 8 using the framework of renormalized solutions to the quasilinear parabolic problems

$$\text{(A.7.1)} \qquad \begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}(d(u)Du + A(u)Du) = \mu & \text{in } (0,T)\times\Omega, \\ u(t=0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \ge 1$, T is a positive real number, Q is the cylinder $(0,T)\times\Omega$ and $(0,T)\times\partial\Omega$ its lateral surface. The coefficients $(d_i(s))_{i=1}^N$ are defined in Chapter 8 and satisfying assumptions $(8.2.1) - (8.2.3)$. The matrix $A(t)$ is defined on $\mathbb{R}$ and is such that

$$\text{(A.7.2)} \qquad A(t) \in C^0(\mathbb{R};\mathbb{R}^{N\times N}),\ A(t)\zeta\cdot\zeta \ge 0 \quad \forall t \in \mathbb{R},\ \forall \zeta \in \mathbb{R}^N.$$

The initial condition $u_0 \in L^1(\Omega)$, $u_0 \le m$ a.e. in Ω and μ is a diffuse measure on Q which does not charges sets of null parabolic capacity. Under these assumptions, it's not clear that problem (A.7.1) admit a distributional solution, since it's necessary to define the vector field $(d_i(u)\frac{\partial u}{\partial x_i})_{i=1}^N$ on Q, and in particular on the subset $\{(t,x) \in Q : u(t,x) = m\}$ where $d_p = +\infty$. In order to have a solution when $\mu \in L^2(Q)$, the definition of $d(u)\nabla u = (d_i(u)\frac{\partial u}{\partial x_i})_{i=1}^N$ must be be coherent with the "a priori" estimates obtained by approximation. Through the truncation on the field $d(s)$ for $s < m - \epsilon$ and for $s > -\frac{1}{\epsilon}$, it is easily to obtain a sequence of approximate solutions $u_\epsilon \in L^2(0,T;H_0^1(\Omega))$. Indeed, coercivity assumption implies that u_ϵ is bounded in $L^2(0,T;H_0^1(\Omega)$. Then one can show that the field $d_\epsilon(u_\epsilon)Du_\epsilon$ is bounded in $(L^2(Q))^N$. Up to a subsequences, it converges weakly in $(L^2(Q))^N$ to an element W of $(L^2(Q))^N$. To obtain the existence of solutions, D. Blanchard and H. Redwane **[BR1]** proposed three notions of solutions in stationary case, to deal with the evolution case recall the following notion of renormalized solutions

Definition A.6. A function u defined on Q is a renormalized solution of (A.7.1) if

$$u \in L^2(0,T;H_0^1(\Omega)), \tag{A.7.3}$$

$$d(u)Du\chi_{\{u<m\}} \in L^2(Q), \tag{A.7.4}$$

$$\text{meas}\{(t,x) \in Q : u(t,x) > m\} = 0, \tag{A.7.5}$$

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \int_{\{m-2\sigma \le u \le m-\sigma\}} [d(u)DuDu + A(u)DuDu]dxdt = \int_{\{u=m\}} \mu \, dxdt \tag{A.7.6}$$

for all $S \in W^{1,\infty}(\mathbb{R})$ such that $S(m) = 0$, we have

$$\begin{aligned} &\frac{\partial S(u)}{\partial t} - \text{div}(S(u)d(u)Du\chi_{\{u<m\}} + S(u)A(u)DuDu) \\ &\quad + S'(u)[d(u)DuDu\chi_{\{u<m\}} + A(u)DuDu] = S(u)\mu \text{ in } \mathcal{D}'(Q) \end{aligned} \tag{A.7.7}$$

Then we can prove the following result

Theorem A.7. *Under assumptions (8.2.1)-(8.2.3) and (A.7.2), problem (A.7.1) admit a renormalized solution in the sense of Definition A.6.*

To conclude, note that we can also give a sufficient additional assumptions on the vector field $d(s)$ and the matrix $A(s)$ in order to propose a comparison principle and then a uniqueness result for solutions of problem (A.7.1).

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Résumé

Nous nous intéressons ici aux équations elliptiques et paraboliques ayant des données peu régulières : des mesures.

Dans le premier chapitre, nous rappelons quelques outils de base, des résultats préliminaires concernant la théorie des problèmes elliptiques et paraboliques à données mesures, nous indiquons la version généralisée des espaces de Lebesgue et de Sobolev et les résultats usuels d'existence. De plus nous introduisons les notations utilisées dans le rapport de thèse.

Le deuxième chapitre traite le cas d'un problème de Dirichlet de forme divergentielle avec une mesure de Radon avec une variation bornée totale et une croissance variable, utilisant des solutions au sens de distributions, nous montrons l'existence et l'unicité des solutions renormalisées en tenant compte de l'hypothèse de Log-Hölder continuité.

Dans le troisième chapitre, nous cherchons la relation entre la capacité parabolique généralisée et les mesures diffuses nécessaire pour avoir l'existence et l'unicité des solutions.

Le quatrième chapitre est consacré à l'étude du comportement asymptotique d'une suite de solutions renormalisées d'un problème parabolique assez général, la difficulté majeure consiste à montrer la convergence forte des troncatures en utilisant des fonctions isolées afin de traiter le terme singulier de la mesure.

Dans le cinquième chapitre, notre approche d'estimation concerne quelques modèles du milieu poreux obtenue par un argument de convolution de la mesure avec une suite régularisante, ainsi le résultat d'existence consiste à montrer la compacité forte des troncatures pour les solutions approchées dans l'espace d'énergie.

Dans le sixième chapitre, nous établissons un résultat similaire pour une classe différente d'opérateurs, appelé équations générales du milieu poreux avec des fonctions non bornées et des mesures générales.

Dans le septième chapitre, nous essayons de montrer la non-stabilité des solutions entropiques pour des inéquations variationnelles avec des données mesures concentrées sur des parties de capacité nulle plus une fonction intégrable.

Le huitième chapitre concerne un résultat d'approximation qui mène à l'existence des solutions renormalisées d'un problème quasi-linéaire de diffusion avec des fonctions qui explosent pour une valeur finie du variable et une mesure générale.

Le rapport termine par une collection de quelques problèmes ouverts et des remarques importantes nécessaires au développement de ce travail

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