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Dédicace

A ma chère mère

Nulle dédicace ne peut exprimer ma profonde affection,

mon immense gratitude pour tous tes sacrifices

et tes prières.

A l'âme de mon père.

A mes deux sœurs et ma femme

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List of Publications

The publications that constitute the basis of the PhD thesis can be found in

Published works and works to appear

[3] M. Ait Hammou and E. Azroul, *Construction of a Topological Degree theory in Generalized Sobolev Spaces*, J. of Univ. Math., 1 (2018), No. 2, 116–129.

[7] M. Ait Hammou, E. Azroul and B. Lahmi, *Topological Degree methods for Partial Differential Operators in Generalized Sobolev Spaces*, Bol. Soc. Paran. Mat. (accepted, article in press).

[9] M. Ait Hammou, E. Azroul and B. Lahmi, *Topological degree methods for a Strongly nonlinear p(x)-elliptic problem*, Rev. Colomb. de Mat. (accepted).

Book chapter

[4] M. Ait Hammou and E. Azroul, *Construction of a Topological Degree Theory in Generalized Sobolev Spaces*, Recent Advances in Intuitionistic Fuzzy Logic Systems, Studies in Fuzziness and Soft Computing 372, Springer Nature Switzerland AG 2019, DOI : 10.1007/978-3-030-02155-9_1.

Submitted works

[5] M. Ait Hammou and E. Azroul, *Nonlinear elliptic problems in weighted variable exponent Sobolev spaces by Topological degree*, Proy. J. of Math.

[8] M. Ait Hammou, E. Azroul and B. Lahmi, *Existence of solutions for* $p(x)$*-Laplacian Dirichlet problem by Topological degree*, Bul. of the Transilvania Univ. of Braşov, S. III : Math., Info., Phys.

[10] M. Ait Hammou, E. Azroul and B. Lahmi, *Existence of solutions for a nonlinear problem involving* $p(x)$*-Laplacian by Topological degree*, Anal. of the Univ. of Craiova, Math. and Comp. Sc. Ser.

[11] M. Ait Hammou, E. Azroul and B. Lahmi, *Nonlinear Elliptic Equations by Topological Degree in Musielak-Orlicz-Sobolev Spaces*, Math. Rep.

[12] M. Ait Hammou, E. Azroul and B. Lahmi, *Existence results for a Dirichlet boundary value problem involving the $p(x)$-Laplacian operator*, Rev. Roumaine Math. Pures Appl.

[13] M. Ait Hammou, E. Azroul and B. Lahmi, *An existence result for a strongly nonlinear parabolic equations with variable nonlinearity*, Azer. J. of Math.

Preprints and works in final phase of preparation

[6] M. Ait Hammou and E. Azroul, *Nonlinear elliptic boundary value problems by Topological degree*.

List of Symbols

Notations

N	natural integer, dimension of the workspace.
$\mathbb{R}$	real line.
$\mathbb{R}^N$	N-dimensional Euclidean space provided with its usual norm.
Ω	open bounded subset of $\mathbb{R}^N$.
$x = (x_1, \cdots, x_N)$	Element of Ω generic point of $\mathbb{R}^N$
$\partial\Omega$	boundary of the set Ω.
∇u	gradient of u defined by $\nabla u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N})$
$p(\cdot)$	variable exponent.
$p'(\cdot)$	conjugate exponent of p, $(\frac{1}{p(x)} + \frac{1}{p'(x)} = 1)$.
a.e	almost everywhere.
D^α	$(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}, ..., \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}})$ with $\alpha = (\alpha_1, ..., \alpha_N)$
$\langle \cdot, \cdot \rangle$	scalar product of $\mathbb{R}^N$, duality between X and X'.
$\rho_{p(\cdot)}(u)$	convex modular, $\rho_{p(x)}(u) = \int_\Omega \lvert u(x)\rvert^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega)$.
$I_{p,\rho}(u)$	$= \int_\Omega \lvert u\rvert^{p(x)} \rho(x) dx, \quad \forall u \in L^{p(x)}(\Omega, \rho)$.
$\lVert u\rVert_X$	norm of the vector u in the space X.
$\lvert\Omega\rvert$	measure of the set Ω.
$\alpha = (\alpha_1, ..., \alpha_n)$	an $n-$tuples or a *multi-index*.
$x = (x_1, \cdot\cdot\cdot, x_N)$	an $n-$tuples of real numbers of $\mathbb{R}^N$.
$dx = dx_1 dx_2 \cdots dx_N$	Lebesgue measure in Ω.
$\Delta_{p(\cdot)} u = \mathrm{div}(\lvert\nabla u\rvert^{p(x)-2} \nabla u)$	$p(x)-$Laplacian operator.

$\hookrightarrow$ ($\hookrightarrow\hookrightarrow$)	continuous (compact) embedding.
$M(.,.) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$	a Musielak-Orlicz function.
$M^{-1}(x,\cdot)$	the inverse of function $M(x,\cdot)$.
$m(x,\cdot)$	the right-hand derivative of $M(x,\cdot)$ for a fixed $x \in \Omega$.
$\overline{M}$	the complementary function to a Musielak-Orlicz function M, $\overline{M}(x,r) = \sup_{s\geq 0}\big(sr - M(x,s)\big)$, for $x \in \Omega, r \geq 0$.
$\varrho_{M,\Omega}$ or ϱ_M	the modular induced by the positive Musielak-Orlicz function M, $\varrho_M(u) = \int_\Omega M(x,\lvert u(x)\rvert)\,dx$.
$\varrho_{1,M}$	the convex modular on $W^1L_M(\Omega)$, $\varrho_{1,M} = \sum_{\lvert\alpha\rvert\leq 1} \varrho_M(D^\alpha u)$.
$\to$ ($\rightharpoonup$)	strong (weak) convergence.
p^- and p^+	$p^- = \operatorname{ess}\inf_\Omega p$ and $p^+ = \operatorname{ess}\sup_\Omega p$.
$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix}$	a complementary system.

Functional spaces and norms

Let u be a measurable function in Ω and $1 \leq p(\cdot) < \infty$.

$C(\Omega)$	space of continuous (real-valued) functions on Ω with the norm $\|f\| = \sup_{x\in\Omega}\|f(x)\|$.
$\mathcal{D}(\Omega)$	class of all infinitely differentiable functions on Ω with compact support endowed with inductive limit topology.
$\mathcal{D}'(\Omega)$	dual space of $C_0^\infty(\Omega)$ (Distribution space).
$C^\infty(\Omega)$	class of infinitely differentiable functions on Ω endowed with topology of uniform convergence on compact sets (smooth functions).
$C^\infty(\overline{\Omega})$	class of $C^\infty(\Omega)$ functions such that all its derivatives can be extended continuously to $\overline{\Omega}$.
$C_0^\infty(\Omega)$	class of all infinitely differentiable functions on Ω with compact support.
$L^{p(\cdot)}(\Omega)$	space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\rho_{p(\cdot)}(u) < \infty$ with the *Luxemburg norm* $\|\|u\|\|_{p(x)} = \inf\{\lambda > 0/\rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}$.
$L^{p'(x)}(\Omega)$	the conjugate space of $L^{p(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$.
$W^{1,p(\cdot)}(\Omega)$	variable exponent Sobolev space $\{u \in L^{p(\cdot)}(\Omega)/\ \nabla u \in (L^{p(\cdot)}(\Omega))^N\}$ with the norm $\\|u\\|_{W^{1,p(x)}(\Omega)} = \\|u\\|_{p(x)} + \\|\nabla u\\|_{p(x)}$.
$W_0^{1,p(x)}(\Omega)$	the closure of $C_0^\infty(\Omega)$ with respect to the norm $\\|\cdot\\|_{W^{1,p(x)}(\Omega)}$ (with the norm $\\|u\\|_{1,p(x)} = \\|\nabla u\\|_{p(x)}$ when $p(.)$ satisfies the log-Hölder continuity condition.)
$W^{-1,p'(x)}(\Omega)$	the dual space of $W_0^{1,p(x)}(\Omega)$ with the norm $\|\|v\|\|_{-1,p'(x)} = \inf\{\|\|v_0\|\|_{p'(x)} + \sum_{i=1}^N \|\|v_i\|\|_{p'(x)}\}$, where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(x)}(\Omega))^N$.
$L^1_{loc}(\Omega)$	the space of locally integrable functions on Ω.
$L^{p(x)}(\Omega, \rho)$	the weighted variable exponent Lebesgue space $\big\{u : \Omega \to \mathbb{R}$ measurable, $\int_\Omega \|u(x)\|^{p(x)}\rho(x)dx < \infty\big\}$ with the norm $\\|u\\|_{p(x),\rho} = \inf\big\{\lambda > 0,\ \int_\Omega \|\frac{u(x)}{\lambda}\|^{p(x)}\rho(x)dx \leq 1\big\}$.
$L^{p'(x)}(\Omega, \rho^*)$	the conjugate space of $L^{p(x)}(\Omega, \rho)$ where $\rho^*(x) = \rho(x)^{1-p'(x)}$.
$W^{1,p(x)}(\Omega, \rho)$	the weighted variable exponent Sobolev space $\{u \in L^{p(x)}(\Omega)/\ \|\nabla u\| \in L^{p(x)}(\Omega, \rho)\}$ with the norm $\\|u\\|_{W^{1,p(x)}(\Omega,\rho)} = \\|u\\|_{p(x)} + \\|\nabla u\\|_{p(x),\rho} \quad \forall u \in W^{1,p(x)}(\Omega, \rho)$.
$W_0^{1,p(x)}(\Omega, \rho)$	the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \rho)$ with the norm $\\|u\\|_{1,p(x),\rho} = \\|\nabla u\\|_{p(x),\rho}$.
$W^{-1,p'(\cdot)}(\Omega, \rho^*)$	the dual space of $W_0^{1,p(\cdot)}(\Omega, \rho)$ with the norm $\\|v\\|_{-1,p'(x),\rho^*} = \inf\Big\{\\|v_0\\|_{p'(x),\rho^*} + \sum_{i=1}^N \\|v_i\\|_{p'(x),\rho^*}\Big\}$. where the infinimum is taken on all possible decompositions $v = v_0 - \text{div } F$.

$K_M(\Omega)$	the Musielak-Orlicz class $\{u:\Omega\to\mathbb{R}$ measurable ; $\varrho_M(u)<\infty\}$.
$L_M(\Omega)$	the Orlicz space, the linear hull of $K_M(\Omega)$ with the *Luxemburg norm* $\|u\|_M=\inf\{k>0\,;\,\int_\Omega M(x,\frac{\lvert u(x)\rvert}{k})\le 1\}$, or the equivalent norm called *Orlicz norm* $\|u\|_{(M)}=\sup\big\{\big\lvert\int_\Omega u(x)v(x)\,dx\big\rvert; v\in K_{\overline{M}}(\Omega), \varrho_{\overline{M}}(v)\le 1\big\}$.
$E_M(\Omega)$	the closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$.
$W^1L_M(\Omega)$	the Musielak-Orlicz-Sobolev space, the space of all $u\in L_M(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_M(\Omega)$ for any α, with $\lvert\alpha\rvert\le 1$ with the norm $\|u\|_{1,(M)}:=\|u\|_{W^1L_M(\Omega)}=\inf\big\{\lambda>0;\varrho_{1,M}(\dfrac{u}{\lambda})\le 1\big\}$, or the equivalent norm $\|u\|_{1,M}:=\|u\|_M+\|\nabla u\|_M$.
$W_0^1L_M(\Omega)$	the norm-closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
$\mathcal{P}(\Omega)$	the set of all measurable function : $p:\Omega\to[1,+\infty]$.
$\mathcal{P}^{log}(\Omega)$	$\mathcal{P}^{log}(\Omega)=\{p\in\mathcal{P}(\Omega):\frac{1}{p}$ is globally log-Hölder continuous $\}$.
$E^{p(\cdot)}(\Omega)$	the closure of the space $L^\infty(\Omega)$ with respect to the Luxemburg norm.
$W^{m,p(\cdot)}(\Omega)$	the space $\{u\in L^{p(\cdot)}(\Omega):D^\alpha u\in L^{p(\cdot)}(\Omega),\lvert\alpha\rvert\le m\}$ with the norm $\lvert\lvert u\rvert\rvert_{m,p(.)}=\sum_{\lvert\alpha\rvert\le m}\lvert\lvert D^\alpha u\rvert\rvert_{p(.)}$.
$H^{m,p(.)}(\Omega)$	the space $\{u\in E^{p(\cdot)}(\Omega):D^\alpha u\in E^{p(\cdot)}(\Omega),\lvert\alpha\rvert\le m\}$ with the norm $\lvert\lvert u\rvert\rvert_{m,p(.)}$.
$W^{-m,p'(\cdot)}(\Omega)$	$W^{-m,p'(\cdot)}(\Omega)=\{F\in\mathcal{D}'(\Omega):F=\sum_{\lvert\alpha\rvert\le m}(-1)^{\lvert\alpha\rvert}D^\alpha f_\alpha,$ where $f_\alpha\in L^{p'(\cdot)}(\Omega)\}$ the dual space of $W^{m,p(.)}(\Omega)$, with the norm $\|F\|_{-m,p'(\cdot)}=\sum_{\lvert\alpha\rvert\le m}\|f_\alpha\|_{p'(\cdot)}$.
$H^{-m,p'(.)}(\Omega)$	$H^{-m,p'(.)}(\Omega)=\{F\in\mathcal{D}'(\Omega):F=\sum_{\lvert\alpha\rvert\le m}(-1)^{\lvert\alpha\rvert}D^\alpha f_\alpha,$ where $f_\alpha\in E^{p'(\cdot)}(\Omega)\}$ the dual space of $H^{m,p(.)}(\Omega)$, with the norm $\|F\|_{-m,p'(\cdot)}$

Functions and intervals

$p(\cdot):\Omega\to[1,+\infty)$	variable exponent.
$\text{sign}(s)=\begin{cases}1 & \text{if } s>0,\\ 0 & \text{if } s=0,\\ -1 & \text{if } s<0.\end{cases}$	sign function.
$\chi_\Omega(x)=\begin{cases}1 & \text{if } x\in\Omega\\ 0 & \text{otherwise}\end{cases}$	characteristic function.
$\varphi_{p(x)}(t):=t^{p(x)}$	for $t\ge 0$, $x\in\Omega$ and $1\le p<\infty$.
$\varphi_\infty(t):=\infty.\chi_{(1,\infty)}(t)$	for $t\ge 0$, $x\in\Omega$.

Introduction Générale

La résolution d'une grande variété de problèmes en analyse (linéaire et non linéaire) se base sur l'étude des équations de la forme

$$F(x) = h. \tag{0.0.1}$$

dans un domaine d'un espace approprié.

Peut-on s'assurer pratiquement qu'il existe au moins une solution à l'équation (0.0.1) ?

Une réponse affirmative dans le cas où F est linéaire continue de $\mathbb{R}^N$ vers $\mathbb{R}^N$ est aquise lorsque le déterminant de F est non nul. De plus, cette solution est unique et stable : si l'on perturbe h ou F (par une application linéaire), on continue d'avoir des solutions qui sont proches de la solution du problème initial. Cependant, lorsque le déterminant de F est nul, il se peut que (0.0.1) admette des solutions pour certains h qui ne sont pas stables : si l'on perturbe légèrement h, (0.0.1) risque de n'avoir aucune solution.

On souhaitait développer un outil jouant, pour des applications non linéaires, le rôle du déterminant pour les applications linéaires : il s'agit du degré topologique, qui indique par sa non nullité que (0.0.1) admette au moins une solution stable. De manière évidente, ce degré topologique dépendra de F et h, mais aussi de l'ensemble sur lequel on cherche les solutions à (0.0.1).

L'idée principale de la construction du degré topologique est de "suivre" les solutions au fur et à mesure que l'on modifie F. On commence par partir d'une fonction F_0 assez simple, pour laquelle il existe des solutions à

$$F_0(x) = h,$$

et on cherche à modifier continûement cette fonction pour arriver à F : en terme précis, on utilise une homotopie approprié entre F_0 et F. Au cours de cette modification, on espére en conserver au moins une solution de (0.0.1) qui appartient au domaine.

Ainsi construit, le degré a des propriétés fondamentales telles que l'existence, la normalisation, l'additivité et l'invariance à l'homotopie.

Pendant les dernières décennies, la théorie du degré topologique s'avère un outil standard, efficace et largement utilisé pour étudier des équations non linéaires et obtenir des théorèmes d'existence et des points fixes ainsi que pour étudier la stabilité. De plus, via le degré topologique, on obtient des résultats concernant les valeurs propres et l'existence de solutions multiples...Vous trouverez un aperçu complet dans [86] (voir aussi [42, 70]).

Un exemple d'utilisation du degré topologique est rencontré dans la résolution d'équations linéaires de convection-diffusion, dites "noncoercitives", de la forme

$$\begin{cases} -\Delta u + div(Vu) = f, & \text{dans } \Omega; \\ u = 0, & \text{sur } \partial\Omega. \end{cases}$$

où $V : \Omega \to \mathbb{R}^N$ est un champ fixé.

Historiquement, La notion de degré topologique a été introduite explicitement par Brouwer [31] en 1912 dans le cas des espaces de dimension finie.

Leray et Schauder [68] ont développé, en 1934, cette théorie pour les opérateurs compacts dans des espaces de Banach de dimention infinie comme une extension de la théorie de degré de Brouwer. Ce qui a permis de traiter des q́uations aux dérivées partielles (EDPs) elliptiques non-linéaires (voir par exemple [40]).

Par la suite, de nombreuses extensions, généralisations et applications de la théorie de degrés ont apparu pour diverses classes d'applications non linéaires et non compactes [37, 57, 60, 61, 71, 83].

L'une des plus importantes généralisations a été réalisé pae Browder [32, 33, 34, 35], qui a étendu, en 1983, ce concept pour les opérateurs non linéaires de type monotone ($f + T$, où f est de classe (S_+) et T est un opérateur monotone maximal) définis d'un espace de Banach réel réflexif dans son espace dual. Sa méthode est basée sur les approximations de Galerkin pour pouvoir appliquer le degré de Brouwer. Cette généralisation a reouvert la voie aux applications de la théorie des degré à une vaste classe d'EDPs non linéaires [32, 57].

J. Berkovits et V. Mustonen introduisent dans [25], en 1986, une nouvelle construction qui se base sur le degré topologique de Leray-Schauder. Pour toute application de classe (S_+), ils ont construit une famille d'approximations, pour laquelle le degré de Leray-Schauder est bien défini, donnant la valeur du degré topologique par passage à la limite. Un outil crucial dans cette construction est le théorème d'injection de Browder et Ton [36] pour les espaces de Banach réflexifs et séparables. Ils ont appliqué ce degré à certains problèmes elliptiques intoduisant un opérateur au différentiel partiel d'ordre $2m$ en forme divergence généralisée

$$Au(x) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, ..., D^m u), \quad x \in \Omega,$$

où Ω est un ouvert de $\mathbb{R}^N \quad (N \geq 2)$. Chaque A_α est une fonction de Carathéodory vérifiant quelques conditions de croissance, de monotonie stricte et de coercivité. Pour $f \in W^{-m,p'}(\Omega)$ donné, ils ont établi l'existence d'une solution faible u dans $W^{-m,p}(\Omega)$ au problème de Dirichlet

$$\begin{cases} Au(x) = f(x), & \text{in } \Omega, \\ D^\alpha u(x) = 0, & \text{on } \partial\Omega \text{ for all } |\alpha| \leq m-1. \end{cases}$$

J. Berkovits [24] introduit une nouvelle extension du degré topologique classique de Leray-Schauder dans un espace de Banach séparable et réflexif. La nouvelle classe d'application pour laquelle ce degré sera construit est obtenue essentiellement en remplaçant la perturbation compacte par une composition d'applications de type monotone (par exemple monotone ou pseudomonotone) : il a reformulé l'équation non linéaire à une équation abstraite de Hammerstein, qui s'écrit, suivant la terminologie de [87], sous la forme :

$$u + ST(u) = 0, \;\; u \in X,$$

où $S : X' \to X$ et $T : X \to X'$ sont des opérateurs de type monotone. Pour éclairer l'utilisation du nouveau degré, il a examiné sa capacité à la résolution des équations de type Hammerstein abstraites en étudiant, dans l'espace de Sobolev $W_0^{1,p}(\Omega)$, l'équation

$$\begin{cases} \sum_{i=1}^N D_i\{|D_i u|^{p-2} D_i u\} = \lambda u(x) + f(x, u, \nabla u), & \text{in } \;\Omega, \\ u(x) = 0, & \text{on } \;\partial\Omega. \end{cases}$$

et des inégalités variationnelles [24, Example 8.3]

L'objectif de ce travail est d'étudier quelques EDPs non linéaires du type elliptiques, en appliquant la théorie du degré topologique et de construire un degré topologique approprié dans les espaces de Sobolev à exposants variables non reflexifs via les systèmes complémentaires. Le cadre naturel dans lequel ces équations peuvent être traitées est celui des espaces Sobolev $W^{1,p(x)}(\Omega)$ et $W^{m,p(x)}(\Omega)$ à exposants variables construits à partir des espaces de Lebesgue $L^{p(x)}(\Omega)$.

Les espaces de Lebesgue à exposants variables sont apparus pour la première fois dans la littérature en 1931 dans l'article de W. Orlicz [76]. Après ce travail, W. Orlicz a abandonné l'étude de ces espaces pour se concentrer sur la théorie des espaces des fonctions qui a pris son nom actuellement "les espaces d'Orlicz" (voir aussi [73]). En 1950, Nakano les a mentionnés explicitement comme un cas particulier des espaces plus généraux discuté dans [74]. La norme de ces espaces (norme de Luxemburg) n'a été presentée qu'en 1979 par Sharapudinov dans [81]. En 1983, Musielak avait étudié les espaces modulaires dans [72] en formulant les espaces $L^{p(x)}$ au moyen de la fonction modulaire $\rho(f) = \int_\Omega \varphi(x, |f(x)|)dx$ en considérant le cas $\varphi(x, f) = |f|^{p(x)}$. Les propriétés de base de ces espaces et des espaces de Sobolev généralisés $W^{m,p(x)}$ telles que la complétude, la séparabilité et la réflexivité ont été étudiées par Kovácik et Rákosník dans [63] en 1991. Ensuite, des résultats d'injections ont été établis par Fan et Zhao [48](voir aussi Edmunds et Rákosník dans [45]). Le problème de la densité a été étudié par Zhikov [90, 93] et par Samko dans [80]. Pour plus de détails sur ces espaces, voir [43] et les références qui y sont mentionnées. Les espaces $L^{p(x)}(\Omega)$ ont été utilisés par la suite par Rajagopal et Růžička dans [77] pour proposer un modèle pour les fluides électroréologiques.

Une classe de problèmes non linéaires avec exposants variables est un nouveau champ de recherche qui reflète un nouveau type de phénomènes physiques. Pour un certain nombre de matériaux ayant des inhomogeniétés, la modélisation en utilisant les espaces de Lebesgue et de Sobolev classiques a démontré sa limitation. Cependant, on s'est interessé de plus en plus pendant la dernière décennie, à l'étude des EDPs et aux problèmes variationnels à $p(\cdot)$-croissance. Cet interêt est motivé par la diversité des domaines de leurs applications :

1) Dans le contexte de la mécanique des fluides, Ladyzhenskaya [64] a souligné que les équations de Navier-Stokes classiques ne suffisent pas à interpréter une classe de fluides (nommés ensuite les fluides non Newtoniens) quand le gradient de la vitesse est assez important et qu'elles doivent être modifiées. Beaucoup d'attention a été donnée à l'étude de ces fluides et surtout les fluides électro-réologiques (une classe importante des fluides nommés aussi fluides intelligentes) qui ont la capacité de changer leurs propriétés mécaniques quand un champ électrique leur est appliqué. En fait, en présence de petites particules (boules) métaliques en grand nombre et sous l'effet d'un champ électrique extérieur $E(x)$, le fluide change d'une façon rapide et réversible ses propriétés mécaniques et surtout la viscosité (leur viscosité peut varier d'un facteur d'ordre 1000 dans des milisecondes). Récemment, Rajagopal et Růžička ont développé dans [77] un modèle très intéressant pour ces fluides prenant en compte la délicate intéraction entre le champ électrique $E(x)$ et le liquide en mouvement :

$$\begin{cases} u_t - \text{div}\, a(x,t,E,D(u)) + \nabla\pi + \nabla u \cdot u = f + \chi^E \nabla E \cdot E \\ \text{rot}\,(E) = 0,\ \text{div}\, E = 0, \\ \text{div}\, u = 0 \end{cases}$$

où χ^E est la susceptibilité diélectrique et π la pression. L'énergie naturelle associée à ce genre de problèmes est donnée par $\int_\Omega |D(u)|^{p(x)}dx$ ou plutôt par

$$\int_\Omega |\nabla(u)|^{p(x)}dx.$$

L'espace d'énergie naturel dans lequel de tels problèmes peuvent être étudiés est l'espace de Sobolev généralisé :

$$W^{1,p(x)}(\Omega) := \{u \in W^{1,1}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega)\}.$$

2) Dans la théorie de la restauration d'images, voir le modèle proposé par Chen, Levine et Rao dans [39].

3) Le phénomène de filtration d'un gaz parfait de pression $\pi = \rho^{\gamma(\theta(x))}$ dans un milieu non homogène poreux de densité ρ où θ est une fonction donnée décrivant les caractéristiques du milieu, est modélisé grâce à la loi de Darcy par l'équation :

$$div(K(x,\pi))\pi^{p(x)}\nabla\pi = h(x,\pi),$$

$h(x,\pi)$ est une matrice diagonale, $p(x) = \frac{1}{\gamma(x)}$ et h la hauteur du barrage poreux étudié en fonction de la position horizontale x. Une étude de ce modèle a été faite en particulier par Antontsev et Shmarev dans [16, 17].

4) Dans la théorie des problèmes de calcul des variations, le problème de minimisation de la fonctionnelle

$$\mathcal{F}(u) = \int_\Omega f(x,\nabla u)dx$$

où le Lagrangien f satisfait les conditions de croissance non standards :

$$|z|^{p(x)} \leq f(x,z) \leq L(1+|z|^{p(x)})$$

où $p(\cdot)$ est continu vérifiant $p(x) > 1$, a été traité par Zhikov dans [90, 91, 92] et par Acerbi et Mingione dans [1, 2].

...

Cette thèse se compose, en plus du premier chapitre qui est consacré aux priliminaires mathématiques concernant un cadre fonctionnel des espaces de Sobolev non standards, les définitions de quelques classes d'opŕateurs et des degrés topologiques de Brower et Berkovits, de deux parties qui peuvent être lues indépendemment :

La première partie, de trois chapitres (2, 3 et 4), est consacrée à l'application du degré topologique de Berkovits pour étudier des problèmes non linéaires elliptiques dans des espaces de Sobolev non standards. Dans le chapitre 2, nous avons étudié le problème $p(x)$-elliptique fortement non linéaire suivant :

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{q(x)-2}u + f(x,u,\nabla u) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega \subset \mathbb{R}^N$, $N \geq 2$, est borné à frontière Lipschitzienne $\partial\Omega$, $p \in C_+(\bar{\Omega})$ satisfaisant la condition du log-Hölder continuité (1.1.7), $q \in C_+(\bar{\Omega})$ tel que $2 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty$ et $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodory satisfaisant une condition de croissance non standard. Nous démontrons que le problème admet au moins une solution au sens faible dans $W_0^{1,p(x)}(\Omega)$.
Dans le chapitre 3, nous étudions le problème elliptique non linéair déegénéré suivant :

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où $p : \overline{\Omega} \to I\!R^+$ est une fonction continue satisfaisant $1 < p^- \leq p(x) \leq p^+ < \infty$ et la condition du log-Hölder continuité (1.1.7). L'opérateurr $Au \equiv -\operatorname{div} a(x, \nabla u)$ est un opérateur de Leray-Lions défini dans $W_0^{1,p(x)}(\Omega, \rho)$, où ρ est une fonction poids satisfaisant quelques conditions d'intégrabilité et f une fonction de Carathéodory satisfaisant une condition de croissance non standard. Nous prouvons l'existence d'une solution faible à ce problème dans $W_0^{1,p(x)}(\Omega, \rho)$.
Le chapitre 4 a été consacré à l'étude du problème non linéaire suivant :

$$\begin{cases} -\operatorname{div} a_1(x, \nabla u) + a_0(x, u) = f(x, u, \nabla u) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où f est une fonction de Carathéodory satisfaisant une condition de croissance, $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ et $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ sont aussi des fonctions de Carathéodory satisfaisant quelques conditions de croissance, de coercivité et de monotonie et qui générent un opérateur de type monotone $-\operatorname{div} a_1(x, \nabla u) + a_0(x, u)$ défini sur $W_0^1 L_\Phi(\Omega)$ à valeurs dans son dual $(W_0^1 L_\Phi(\Omega))'$. Ici, Φ est une fonction Musielak-Orlicz satisfaisant quelques conditions, notament la condition Δ_2 qui assure la réflexivité de ces espaces. On a démontré qu'il existe au moins une solution faible à ce problème.

La deuxième partie de cette thèse est constituée de deux chapitres (5 et 6).
Le chapitre 5 est consacré à la construction d'un degré topologique en utilisant un système complémentaire des espaces de Sobolev généralisés non nécessairement réflexifs, où l'exposant variable $p \in \mathcal{P}^{log}(\Omega)$ satisfait $1 \leq p^- \leq p^+ < \infty$ et à élaborer quelques prpriétés de ce degré.
Dans le chapitre 6, on applique le degré topologique construit au chapitre 5 pour prouver l'existence d'une solution au problème elliptique non linéaire suivant :

$$\begin{cases} A(u) = f & \text{in} \quad \Omega \\ D^\alpha u(x) = 0 & \text{on} \quad \partial\Omega \text{ for } |\alpha| \leq m - 1. \end{cases}$$

où $Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, ..., \nabla^m u)$ est un opérateur différentiel partiel de forme divergence générale et $f \in W^{-m,p'(.)}(\Omega)$ avec $\Omega \subset \mathbb{R}^N$ un ouvert borné de frontière Lipschitzienne $\partial\Omega$ et $p(.)$

vérifie la condition du log-Hölder continuité tel que $1 < p^- \leq p(x) \leq p^+ < \infty$. Dans ce chapitre, on a présenté deux opérateurs qui peuvent être utilisés comme opérateurs normalisateurs.

Chapitre 1

Preliminaries

1.1 Variable Lebesgue and Sobolev spaces

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with a Lipschitz boundary denoted by $\partial\Omega$. We first recall some preliminary definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. More details can be found in [43, 45, 50, 63, 88].
Denote

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) | \inf_{x\in\bar{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ := max\{h(x), x \in \bar{\Omega}\}, h^- := min\{h(x), x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

endowed with *Luxemburg norm*

$$||u||_{p(x)} = \inf\{\lambda > 0/\rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega).$$

$(L^{p(x)}(\Omega), ||\ .\ ||_{p(x)})$ is a Banach space [63, Theorem 2.5], separable and reflexive [63, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, Hölder inequality holds [63, Theorem 2.1]

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) ||u||_{p(x)}||v||_{p'(x)} \leq 2||u||_{p(x)}||v||_{p'(x)}. \tag{1.1.1}$$

Notice that if (u_n) and $u \in L^{p(x)}(\Omega)$ then the following relations hold true (see [50])

$$||u||_{p(x)} < 1(= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1(= 1; > 1),$$

$$||u||_{p(x)} > 1 \quad \Rightarrow \quad ||u||_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq ||u||_{p(x)}^{p^+}, \tag{1.1.2}$$

$$||u||_{p(x)} < 1 \quad \Rightarrow \quad ||u||_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq ||u||_{p(x)}^{p^-}, \tag{1.1.3}$$

$$\lim_{n\to\infty} ||u_n - u||_{p(x)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{1.1.4}$$

From (1.1.2) and (1.1.3), we can deduce the inequalities

$$||u||_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{1.1.5}$$

$$\rho_{p(x)}(u) \leq ||u||_{p(x)}^{p^-} + ||u||_{p(x)}^{p^+}. \tag{1.1.6}$$

If $p_1, p_2 \in C_+(\bar{\Omega}), p_1(x) \leq p_2(x)$ for any $x \in \bar{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ \ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is a Banach space under the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p(x)}(\Omega)}$. If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x, y \in \bar{\Omega}, x \neq y$ with $|x - y| \leq \frac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x - y|}\ \ , \tag{1.1.7}$$

then we have the Poincaré inequality (see [55, 80]), i.e. the exists a constant $C > 0$ depending only on Ω and the function p such that

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \forall u \in W_0^{1,p(.)}(\Omega). \tag{1.1.8}$$

In particular, the space $W_0^{1,p(x)}(\Omega)$ has a norm $||\ .\ ||_{1,p(x)}$ given by

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.

In addition, we have the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$(see [63]).

The space $(W_0^{1,p(x)}(\Omega), ||\,.\,||_{1,p(x)})$ is a Banach space, separable and reflexive (see [50, 63]).
The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$||v||_{-1,p'(x)} = \inf\{||v_0||_{p'(x)} + \sum_{i=1}^{N} ||v_i||_{p'(x)}\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(x)}(\Omega))^N$.

1.2 Weighted Lebesgue and Sobolev spaces with variable exponents

In what follows, we recall some definitions and basic properties of weighted Lebesgue and Sobolev spaces with variable exponents (more detailed description can be found in [18]).
Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary $\partial\Omega$ and $p\ :\ \overline{\Omega} \to I\!R^+$ is a continuous function satisfying

$$1 < p^- \leq p(x) \leq p^+ < +\infty, \tag{1.2.1}$$

and the log-Hölder continuity condition (1.1.7). Let ρ be a function defined on Ω, ρ is called a weight function if it is a measurable and strictly positive a.e. in Ω. Let introduce the integrability conditions used on the framework of weighted variable Lebesgue and Sobolev spaces

$$(H_1) : \rho \in L^1_{loc}(\Omega)$$
$$(H_2) : \rho^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$$

We define the weighted variable exponent Lebesgue space by

$$L^{p(x)}(\Omega, \rho) = \big\{u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |u(x)|^{p(x)}\rho(x)dx < \infty\big\}.$$

The space $L^{p(x)}(\Omega, \rho)$ endowed with the norm :

$$\|u\|_{p(x),\rho} = \inf\big\{\lambda > 0,\ \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\rho(x)dx \leq 1\big\}$$

is a uniformly convex Banach space, thus reflexive. We denote by $L^{p'(x)}(\Omega, \rho^*)$ the conjugate space of $L^{p(x)}(\Omega, \rho)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, and $\rho^*(x) = \rho(x)^{1-p'(x)}$.
As in [48] we can prove the following proposition

Proposition 1.2.1 *1. For any $u \in L^{p(x)}(\Omega, \rho)$ and $v \in L^{p'(x)}(\Omega, \rho^*)$, we have the Hölder inequality*

$$|\int_\Omega uv dx| \leq (\frac{1}{p_-} + \frac{1}{p'_-})\|u\|_{p(x),\rho}\|v\|_{p'(x),\rho^*}.$$

2. *For all p_1, p_2 continuous on $\overline{\Omega}$ such that $p_1(x) \leq p_2(x)$ a.e $x \in \overline{\Omega}$, we have* $L^{p_2(x)}(\Omega, \rho) \hookrightarrow L^{p_1(x)}(\Omega, \rho)$ *and the embedding is continuous.*

Let us denote

$$I_{p,\rho}(u) = \int_{\Omega} |u|^{p(x)} \rho(x) dx, \quad \forall u \in L^{p(x)}(\Omega, \rho).$$

By taking $I_{p,\rho}(u) = I(\rho^{\frac{1}{p(x)}} u)$, where $I(u) = \int_{\Omega} |u|^{p(x)} dx$ and $\|\rho^{\frac{1}{p(x)}} u\|_{p(x)} = \|u\|_{p(x),\rho}$, we can prove the following result as a consequence of the corresponding one in [48].

Proposition 1.2.2 *For each $u \in L^{p(x)}(\Omega, \rho)$,*

1. $\|u\|_{p(x),\rho} < 1$ *(resp. = 1 or > 1)* $\Leftrightarrow I_{p,\rho}(u) < 1$ *(resp. = 1 or > 1),*
2. $\|u\|_{p(x),\rho} > 1 \Rightarrow \|u\|^{p-}_{p(x),\rho} \leq I_{p,\rho}(u) \leq \|u\|^{p+}_{p(x),\rho}$,
   $\|u\|_{p(x),\rho} < 1 \Rightarrow \|u\|^{p+}_{p(x),\rho} \leq I_{p,\rho}(u) \leq \|u\|^{p-}_{p(x),\rho}$,
   $I_{p,\rho}(u) \leq \|u\|^{p+}_{p(x),\rho} + \|u\|^{p-}_{p(x),\rho}$,
3. $\|u\|_{p(x),\rho} \to 0 \Leftrightarrow I_{p,\rho}(u) \to 0$ *and* $\|u\|_{p(x),\rho} \to \infty \Leftrightarrow I_{p,\rho}(u) \to \infty$.

We define the weighted variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega, \rho) = \{u \in L^{p(x)}(\Omega) / \ |\nabla u| \in L^{p(x)}(\Omega, \rho)\}.$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\rho)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x),\rho} \quad \forall u \in W^{1,p(x)}(\Omega, \rho).$$

We denote by $W_0^{1,p(x)}(\Omega, \rho)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \rho)$.
We will use the following result of compact imbedding which can be proved in a similar manner to that of Theorem 8.4.2. in [43] (see also [63])

Proposition 1.2.3 $W_0^{1,p(x)}(\Omega, \rho) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$.

The dual space of $W_0^{1,p(\cdot)}(\Omega, \rho)$, denoted $W^{-1,p'(\cdot)}(\Omega, \rho^*)$, is equipped with the norm

$$\|v\|_{-1,p'(x),\rho^*} = \inf \left\{ \|v_0\|_{p'(x),\rho^*} + \sum_{i=1}^{N} \|v_i\|_{p'(x),\rho^*} \right\}$$

where the infinimum is taken on all possible decompositions $v = v_0 - \text{div } F$.

Remark 1.2.1 We can see following [79, Theorem 3] that the the Poincaré inequality holds for the weighted Sobolev spaces $W_0^{1,p(\cdot)}(\Omega, \rho)$. In particular, this space has a norm $\| \cdot \|_{1,p(x),\rho}$ given by

$$\|u\|_{1,p(x),\rho} = \|\nabla u\|_{p(x),\rho} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega, \rho),$$

which equivalent to $\| \cdot \|_{W^{1,p(x)}(\Omega)}$.

1.3 Musielak and Musielak-Orlicz-Sobolev spaces

Standard references on Musielak-Orlicz-Sobolev spaces and their properties include [46, 58, 72] and references therein.

Definition 1.3.1 Let Ω be an open subset of $\mathbb{R}^N$. A function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a Musielak-Orlicz function if

1. $M(x, \cdot)$ is an N-function, i.e. convex, nondecreasing, continuous, $M(x,0) = 0, M(x,t) > 0 (\forall t > 0),\ \lim_{t\to 0^+} \sup_{x\in\Omega} \dfrac{M(x,t)}{t} = 0$ and $\lim_{t\to +\infty} \inf_{x\in\Omega} \dfrac{M(x,t)}{t} = +\infty,$
2. $M(\cdot, t)$ is a measurable function,

For each $x \in \Omega$, the inverse of function $M(x, \cdot)$ is denoted by $M_x^{-1}(\cdot)$ or for simplicity $M^{-1}(x, \cdot)$ and then $M^{-1}(x, M(x, s)) = s$ and $M(x, M^{-1}(x, s)) = s$ for all $s \geq 0$.

Remark 1.3.1 M admits the representation

$$M(x,t) = \int_0^t m(x,s)\, ds, \text{ for all } t \geq 0,$$

where $m(x, \cdot)$ is the right-hand derivative of $M(x, \cdot)$ for a fixed $x \in \Omega$.
We recall that for every x in Ω, the function $m(x, \cdot)$ is a right-continuous and nondecreasing verifying for all $s \geq 0 : m(x, 0) = 0$, $m(x, s) > 0$ for $s > 0$, $\lim_{s\to+\infty} \inf_{x\in\Omega} m(x, s) = +\infty$ and
$M(x, s) \leq sm(x, s) \leq M(x, 2s)$.

The complementary function $\overline{M}$ to a Musielak-Orlicz function M is defined as follows :

$$\overline{M}(x,r) = \sup_{s\geq 0} \big(sr - M(x,s)\big), \qquad \text{for } x \in \Omega, r \geq 0.$$

Note that $\overline{M}$ is a Musielak-Orlicz function which admits a similar representation where $\overline{m}$ is defined as above or by

$$\overline{m}(x,s) = \sup\{\delta;\ m(x,\delta) \leq s\}.$$

We recall *Young's inequality*

$$r \cdot s \leq M(x,s) + \overline{M}(x,r), \quad \forall r, s \in \mathbb{R}^+, x \in \Omega,$$

Note that when $\overline{M}$ satisfy the Δ_2-condition, a variant of Young's inequality holds, i.e.,

$$r \cdot s \leq \varepsilon M(x,s) + c(\varepsilon)\overline{M}(x,r), \quad \forall r, s \in \mathbb{R}^+, x \in \Omega,$$

where $\varepsilon \in]0,1[$ and $c(\varepsilon)$ a constant depending of ε.

For $u : \Omega \to \mathbb{R}$ measurable function, we define the modular $\varrho_{M,\Omega}$ or ϱ_M induced by the positive Musielak-Orlicz function M as

$$\varrho_M(u) = \int_\Omega M(x, |u(x)|)\, dx$$

Let us consider the *Musielak-Orlicz class*

$$K_M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\,;\, \varrho_M(u) < \infty\}.$$

The *Orlicz space* $L_M(\Omega)$ is defined as the linear hull of $K_M(\Omega)$ and it is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_M = \inf\{k > 0\,;\, \int_\Omega M(x, \frac{|u(x)|}{k}) \leq 1\}.$$

Or the equivalent norm called *Orlicz norm*

$$\|u\|_{(M)} = \sup\big\{\big|\int_\Omega u(x)v(x)\, dx\big|; v \in K_{\overline{M}}(\Omega), \varrho_{\overline{M}}(v) \leq 1\big\}.$$

One has a *Hölder's type inequality* : if $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\big|\int_\Omega u(x)v(x)\, dx\big| \leq 2\|u\|_M \|v\|_{\overline{M}}.$$

The closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. It is a separable space and $(E_{\overline{M}}(\Omega))' = L_M(\Omega)$. Generally $K_M(\Omega) \subset L_M(\Omega)$ but we can obtain $E_M(\Omega) = L_M(\Omega) = K_M(\Omega)$ if and only if M satisfies the Δ_2-condition, i.e. there is a constant $k > 1$ independent of $x \in \Omega$ and a nonnegative function $h \in L^1(\Omega)$

$$M(x, 2s) \leq kM(x, s) + h(x), \qquad \text{for all } s \geq 0, \text{ a.e. } x \in \Omega.$$

Note also that under this condition, the space $L_M(\Omega)$ is reflexive.

Let M and P two Musielak-Orlicz functions, $M \preceq P$ means that M is weaker than P, i.e. there is two positive constants k_1 and k_2 and a nonnegative function $H \in L^1(\Omega)$ such that

$$M(x, s) \leq k_1 P(x, k_2 s) + H(x), \qquad \text{for all } s \geq 0, \text{ a.e. } x \in \Omega.$$

Remark 1.3.2 *[58, 72]*

Let M and P two Musielak-Orlicz functions such that $M \preceq P$. Then $\overline{P} \preceq \overline{M}$, $L_P(\Omega) \hookrightarrow L_M(\Omega)$ and $L_{\overline{M}}(\Omega) \hookrightarrow L_{\overline{P}}(\Omega)$.

We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_M(\frac{u_n - u}{\lambda}) \to 0, \quad \text{when } n \to +\infty.$$

In any Musielak-Orlicz space, norm convergence implies the modular convergence and the modular convergence implies the weak convergence.

Proposition 1.3.1 *[51, 58, 72] Let M be a Museilak-Orlicz function satisfy Δ_2-condition. Let $u \in L_M(\Omega)$ and $(u_n)_n \subset L_M(\Omega)$. Then the following assertions hold.*

1. $\displaystyle\int_\Omega M(x, u_n)\, dx > 1$ (*resp* $= 1; < 1$) $\Leftrightarrow \|u\|_M > 1$ (*resp* $= 1; < 1$),
2. $\displaystyle\int_\Omega M(x, u_n)\, dx \underset{n\to\infty}{\to} 0$ (*resp* $= 1; +\infty$) $\Leftrightarrow \|u_n\|_M \underset{n\to\infty}{\to} 0$ (*resp* $= 1; +\infty$),
3. $u_n \underset{n\to\infty}{\to} u$ *in* $L_M(\Omega) \Rightarrow \displaystyle\int_\Omega M(x, u_n)\, dx \underset{n\to\infty}{\to} \int_\Omega M(x, u)\, dx$,
4. $\|u\|_M \leq \varrho_M(u) + 1$,
5. $m(\cdot, u(\cdot)) \in L_{\overline{M}}(\Omega)$ *(the function m is defined in remark 1.3.1).*

The Musielak-Orlicz-Sobolev space $W^1L_M(\Omega)$ is the space of all $u \in L_M(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_M(\Omega)$ for any α, with $|\alpha| \leq 1$. Let $\varrho_{1,M} = \sum_{|\alpha|\leq 1} \varrho_M(D^\alpha u)$ the convex modular on $W^1L_M(\Omega)$. The space $W^1L_M(\Omega)$ equipped with the norm

$$\|u\|_{1,(M)} := \|u\|_{W^1L_M(\Omega)} = \inf\big\{\lambda > 0; \varrho_{1,M}(\frac{u}{\lambda}) \leq 1\big\},$$

or the equivalent norm

$$\|u\|_{1,M} := \|u\|_M + \|\nabla u\|_M.$$

This space is a Banach space if and only if there is a constant c such that $\inf_{x\in\Omega} M(x, 1) > c$ (see [72]). The space $W_0^1L_M(\Omega)$ is defined as the norm-closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. Moreover if this condition is satisfied, then $W^1L_M(\Omega)$ and $W_0^1L_M(\Omega)$ are separable Banach spaces and

$$W_0^1L_M(\Omega) \hookrightarrow W^1L_M(\Omega) \hookrightarrow W^{1,1}(\Omega).$$

We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_{1,M}(\frac{u_n - u}{\lambda}) \to 0, \quad \text{when } n \to +\infty.$$

Suppose also that

$$\lim_{t\to 0} \int_t^1 \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau < \infty, \quad \lim_{t\to\infty} \int_1^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau = \infty. \tag{1.3.1}$$

With (1.3.1) satisfied, we define the *Sobolev conjugate* M_* of M as the reciprocal function of F with respect to t where

$$F(x,t) = \int_0^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau, t \geq 0.$$

Proposition 1.3.2 *[22] If the Musielak-Orlicz function M satisfies* (1.3.1)*, then*

$$W_0^1 L_M(\Omega) \hookrightarrow L_{M_*}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbeddings

$$W_0^1 L_M(\Omega) \hookrightarrow\hookrightarrow L_P(\Omega_0)$$

exist and are compact for any Musielak-Orlicz function P increasing essentially more slowly than M near infinity (see proof of Theorem 4. in [22] for more informations).

We have the following result :

Lemma 1.3.1 *Let Ω be a bounded domain in $\mathbb{R}^N$. Let Φ a Musielak-Orlicz function locally integrable satisfy Δ_2-condition such that $\inf_{x\in\Omega} \Phi(x,1) = c > 0$. If $(u_n)_n \subset L_\Phi(\Omega)$ with $u_n \to u$ in $L_\Phi(\Omega)$, then there exists $\widetilde{w} \in L_\Phi(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that :*

$$|u_{n_k}(x)| \leq \widetilde{w}(x), \quad \textit{and} \quad u_{n_k}(x) \to u(x) \textit{ a.e. in } \Omega.$$

Proof. Let $(u_n) \subset L_\Phi(\Omega)$ such that $u_n \to u$ in $L_\Phi(\Omega)$, we can suppose that $\|(u_n - u)\|_\Phi \leq \frac{1}{2}$, then by proposition 1.3.1

$$\int_\Omega \Phi(x, 2(u_n(x) - u(x))\, dx \leq 2\|(u_n-u)\|_\Phi \int_\Omega \Phi(x, \frac{u_n(x)-u(x)}{\|(u_n-u)\|_\Phi})\, dx.$$
$$\leq 2\|(u_n-u)\|_\Phi.$$

Therefore $\|\varrho_\Phi(u_n - u)\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. On other hand, since Ω has a finite measure, the continuous embedding $L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$ hold (by using the generalized Hölder's inequality) then $u_n \to u$ in $L^1(\Omega)$. We deduce that there exists $w \in L^1(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that $u_{n_k}(x) \to u(x)$ a.e. in Ω and $\Phi(x, u_{n_k}(x) - u(x)) \leq w(x)$ a.e. in Ω. Since Φ_x^{-1} is a nondecreasing function, we obtain

$$\mid u_{n_k}(x) \mid \leq \mid u(x) \mid + \Phi^{-1}(x, w(x)).$$

Let $\widetilde{w}(x) = \mid u(x) \mid + \Phi^{-1}(x, w(x))$, then

$$\int_\Omega \Phi(x, \widetilde{w}(x))\, dx \leq \frac{1}{2}\int_\Omega \Phi(x, 2 \mid u(x) \mid) + \int_\Omega w(x)\, dx.$$

Thus $\widetilde{w} \in K_\Phi(\Omega) = L_\Phi(\Omega)$. □

1.4 Some classes of mappings

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \, , \, . \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Let Y be a real Banach space. We recall that a mapping $F : \Omega \subset X \to Y$ is :

(i) *bounded,* denote $F \in (BD)$, if it takes any bounded set into a bounded set.

(ii) *continuous,* denote $F \in (CONT)$, if the conditions $\{u_n\} \subset \Omega$, $u \in \Omega$ and $u_n \to u$ imply that $F(u_n) \to F(u)$.

(iii) *demicontinuous,* denote $F \in (DC)$, if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.

(iv) *finitely continuous,* denote $F \in (FC)$, if the restriction of the mapping F to any finite dimensional supspace $X_0 \subset X$ is continuous from the topology of X_0 to the weak topology of Y .

(v) *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be

(i) *of class* (S_+)*,* denote $F \in (S_+)$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

(ii) *quasimonotone,* denote $F \in (QM)$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

(iii) *strongly quasibounded* , denote $F \in (QB)$, if the conditions $\{u_n\} \subset \Omega$ is bounded and $\langle F(u_n), u_n - \bar{u}\rangle$ is bounded from some $\bar{u} \in X$ imply that $\{F(u_n)\}$ is bounded in X^*.

(iv) *pseudomonotone*, $F \in (PM)$, if the conditions $\{u_n\} \subset \Omega$, $u_n \rightharpoonup u$ in X , $F(u_n) \rightharpoonup \chi$ in X^* and $limsup_{n\to\infty}\langle F(u_n), u_n\rangle \leq \langle \chi, u\rangle$ imply that $u \in \Omega$, $\chi = F(u)$ and $\langle F(u_n), u_n\rangle \to \langle F(u), u\rangle$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

1.5 Brouwer and Berkovits degree theory

1.5.1 An outline of Brouwer degree theory

Theorem 1.5.1 *Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets G of X, consider continuous mappings $f : \bar{G} \to Y$ and points y_0 in Y such that $y_0 \notin f(\partial G)$. Then to each such triple (f, G, y_0), there corresponds an integer $d(f, G, y_0)$ having the following properties :*

(a) *Existence : if* $d(f, G, y_0) \neq 0$, *then* $y_0 \in f(G)$,

(b) *Additivity : if* $f : \bar{G} \to Y$ *is a continuous map with* G *a bounded open set in* X *and* G_1, G_2 *are a pair of disjoint open subsets of* G *such that* $y_0 \notin f(\bar{G} \setminus (G_1 \cup G_2))$, *then*

$$d(f, G, y_0) = d(f, G_1, y_0) + d(f, G_2, y_0),$$

(c) *Invariance under homotopy : Let* G *be a bounded open set in* X, *and consider a continuous homotopy* $\{f_t : 0 \leq t \leq 1\}$ *of maps of* $\bar{G}$ *into* Y. $\{y_t : 0 \leq t \leq 1\}$ *be a continuous curve in* Y *such that* $y_t \notin f_t(\partial G)$ *for any* $t \in [0, 1]$, *then* $d(f_t, G, y_t)$ *is constant in* t *on* $[0, 1]$,

(d) *Normalization : If* f_0 *is the identity map of* X *onto* Y, *then for every bounded open* G *and* $y_0 \in f_0(G)$ *then*

$$d(f_0, G, y_0) = 1.$$

Theorem 1.5.2 *The degree function* $d(f, G, y0)$ *is uniquely determined by the four conditions of Theorem 1.5.1.*

Remark 1.5.1 *Theorem 1.5.1 is an appropriately formalized version of the properties of the classical Brouwer degree. Theorem 1.5.2 contains an observation made independently in 1972 and 1973 by Fuhrer [51] and Amann and Weiss [15], respectively.*

1.5.2 An outline of Berkovits degree theory

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators :

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satifies condition } (S_+)\}, \\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and satifies condition } (S_+)_T\}, \\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \text{ is demicontinuous and satifies condition } (S_+)_T\}, \\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 1.5.1 *[24, Lemmas 2.2 and 2.4] Suppose that* $T \in \mathcal{F}_1(\bar{G})$ *is continuous and* $S : D_S \subset X^* \to X$ *is demicontinuous such that* $T(\bar{G}) \subset D_s$, *where* G *is a bounded open set in a real reflexive Banach space* X. *Then the following statements are true :*

(i) *If* S *is quasimonotone, then* $I + SoT \in \mathcal{F}_T(\bar{G})$, *where* I *denotes the identity operator.*

(ii) *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$*

Definition 1.5.1 *Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. The affine homotopy $H : [0,1] \times \bar{G} \to X$ defined by*

$$H(t,u) := (1-t)Fu + tSu \text{ for } (t,u) \in [0,1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 1.5.2 *[24] The above affine homotopy satisfies condition $(S_+)_T$.*

We introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [24].

Theorem 1.5.3 *There exists a unique degree function*

$$d : \{(F,G,h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties

1. *(Existence) if $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
2. *(Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$

3. *(Homotopy invariance) If $H : [0,1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$,then the value of $d(H(t,.), G, h(t))$ is constant for all $t \in [0,1]$.*
4. *(Normalization) For any $h \in G$, we have $d(I,G,h) = 1$.*

Première partie

Application of Berkovits degree to nonlinear elliptic problems

Chapitre 2

Topological degree methods for a Strongly nonlinear $p(x)$-elliptic problem

Abstract

This chapter is devoted to study the existence of weak solutions for the strongly nonlinear $p(x)$-elliptic problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda |u|^{q(x)-2}u + f(x,u,\nabla u) & ,x \in \Omega \quad , \\ u = 0 & ,x \in \partial\Omega \quad . \end{cases}$$

Our technical approach is based on the recent Berkovits topological degree.

2.1 Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. The specific attention accorded to such kind of problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomenon which arise in elastic mechanics or electrorheological fluids (sometimes referred to as "smart fluids")(see [78, 89]). Many results have been obtained for this kind of problems, for instance we here cite [27, 28, 29, 41, 47, 48].

We consider the following nonlinear $p(x)$-elliptic problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda |u|^{q(x)-2}u + f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

where $-\Delta_{p(x)}(u) = -div(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p(\cdot), q(\cdot) \in C(\bar{\Omega})$ and λ is a real parameter. We assume also that $p(\cdot)$ is log-Hölder continuous function and

$$2 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty.$$

For $\lambda = 0$ and f independent of ∇u, Fan and Zhang (in [48]) presents several sufficient conditions for the existence of solutions for the problem. Their discussion is based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. The same problem is studied after by P.S. Iliaş (in [59]) who discusses sufficient conditions which allow to use variational and topological methods to prove the existence of weak solutions.

For $f \equiv 0$ and $p(.) = q(.)$, X. Fan and al. (in [49]) study the eigenvalues of the problem. The present some sufficient conditions for $\inf \Lambda = 0$ and for $\inf \Lambda > 0$, respectively where Λ is the set of eigenvalues.

R. ALsaedi (in [14]) establishes sufficient conditions for the existence of nontrivial weak solutions for the following problem :

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u + \mu |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proofs combine the Ekeland variational principle, the mountain pass theorem and energy arguments.

In this chapter, we will generalize these works, by proving, under conditions on the functions p and q and a suitable growth condition of f, the existence of weak solutions for the problem (2.1.1). Our technical approach is based on the recent Berkovits topological degree.

The chapter is divided into three sections. In the second section, we introduce some important properties of $p(x)$-Laplacian operator. Finally, in the third section, we give the assumptions and our main results concerning the weak solutions of problem (2.1.1).

2.2 Properties of $p(x)-$Laplacian operator

Next, we discuss the $p(x)-$Laplacian operator

$$-\Delta_{p(x)} u := -div(|\nabla u|^{p(x)-2}\nabla u).$$

Consider the following functional :

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [38]), $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and the $p(x)-$Laplacian operator is the derivative operator of J in the weak sense.

We denote $L = J' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, then

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Theorem 2.2.1 *[38, Theorem 3.1]*

(i) $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator ;*

(ii) L *is a mapping of class* (S_+) *;*

(iii) L *is a homeomorphism.*

2.3 Assumptions and Main Results

In this section, we study the strongly nonlinear problem (2.1.1) based on the degree theory in Section 2, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\bar{\Omega})$ satisfies the log-Hölder continuity condition (1.1.7), $q \in C_+(\bar{\Omega})$, $2 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a real-valued function such that :

(f_1) f satisfies the Carathéodory condition, that is, $f(., \eta, \zeta)$ is measurable on Ω for all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^n$ and $f(x, ., .)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for a.e. $x \in \Omega$.

(f_2) f has the growth condition

$$|f(x, \eta, \zeta)| \leq c(k(x) + |\eta|^{r(x)-1} + |\zeta|^{r(x)-1})$$

for a.e. $x \in \Omega$ and all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^n$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$, $k(x) \geq 0$ and $r \in C_+(\bar{\Omega})$ with $2 < r^- \leq r(x) \leq r^+ < p^-$.

Definition 2.3.1 *We call that* $u \in W_0^{1,p(x)}(\Omega)$ *is a weak solution of (2.1.1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega (\lambda|u|^{q(x)-2}u + f(x, u, \nabla u))v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

Remark 2.3.1 *Note that* $\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \langle Lu, v\rangle$ *as defined in section 2,* $\lambda|u|^{q(x)-2}u \in L^{p'(x)}(\Omega)$ *and* $f(x, u, \nabla u) \in L^{p'(x)}(\Omega)$ *under* $u \in W_0^{1,p(x)}(\Omega)$ *and the given hypotheses about the exponents* p, q *and* r *and condition* (f_2) *because :* $k \in L^{p'(x)}(\Omega)$, $\alpha(x) = (q(x) - 1)p'(x) \in C_+(\bar{\Omega})$ *with* $\alpha(x) < p(x)$ *and* $\beta(x) = (r(x) - 1)p'(x) \in C_+(\bar{\Omega})$ *with* $\beta(x) < p(x)$*. Then, we can conclude by the continuous embedding* $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ *and* $L^{p(x)} \hookrightarrow L^{\beta(x)}$*.*
Since $v \in L^{p(x)}(\Omega)$*, we have* $(\lambda|u|^{q(x)-2}u + f(x, u, \nabla u))v \in L^1(\Omega)$*. Then the integral* $\int_\Omega (\lambda|u|^{q(x)-2}u + f(x, u, \nabla u))v dx$ *exist.*

Lemma 2.3.1 *Under assumptions* (f_1) *and* (f_2)*, the operator*
$S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *setting by*

$$\langle Su, v\rangle = -\int_\Omega (\lambda|u|^{q(x)-2}u + f(x,u,\nabla u))v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega)$$

is compact.

Proof. **Step 1**

Let $\phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be the operator defined by

$$\phi u(x) := -\lambda|u(x)|^{q(x)-2}u(x) \text{ for } u \in W_0^{1,p(x)}(\Omega) \text{ and } x \in \Omega.$$

It's obvious that ϕ is continuous. We prove that ϕ is bounded.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by the inequalities (1.1.5) and (1.1.6) that

$$\begin{aligned}
\|\phi u\|_{p'(x)} &\leq \rho_{p'(x)}(\phi u) + 1 \\
&= \int_\Omega |\lambda|u|^{q(x)-1}|^{p'(x)} \, dx + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})\rho_{\alpha(x)}(u) + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})(\|u\|^{\alpha^-}_{\alpha(x)} + \|u\|^{\alpha^+}_{\alpha(x)}) + 1,
\end{aligned}$$

Then, by the continuous embedding $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and the Poincaré inequality (1.1.8), we have

$$\|\phi u\|_{p'(x)} \leq const(\|u\|^{\alpha^-}_{1,p(x)} + \|u\|^{\alpha^+}_{1,p(x)}) + 1$$

This implies that ϕ is bounded on $W_0^{1,p(x)}(\Omega)$.

Step 2

Let $\psi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\psi u(x) := -f(x,u,\nabla u) \text{ for } u \in W_0^{1,p(x)}(\Omega) \text{ and } x \in \Omega.$$

We prove that ψ is bounded and continuous.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by the growth condition (f_2), the inequalities (1.1.5) and (1.1.6) that

$$\begin{aligned}
\|\psi u\|_{p'(x)} &\leq \rho_{p'(x)}(\phi u) + 1 \\
&= \int_\Omega |f(x,u(x),\nabla u(x)|^{p'(x)} \, dx + 1 \\
&\leq const(\rho_{p'(x)}(k) + \rho_{\beta(x)}(u) + \rho_{\beta(x)}(\nabla u)) + 1 \\
&\leq const(\|k\|^{p'^-}_{p'(x)} + \|k\|^{p'^+}_{p'(x)} + \|u\|^{\beta^-}_{\beta(x)} + \|u\|^{\beta^+}_{\beta(x)} + |\nabla u|^{\beta^-}_{\beta(x)} + |\nabla u|^{\beta^+}_{\beta(x)}) + 1,
\end{aligned}$$

Then, by the continuous embedding $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and the Poincaré inequality (1.1.8), we have

$$\|\psi u\|_{p'(x)} \leq const(\|k\|^{p'^-}_{p'(x)} + \|k\|^{p'^+}_{p'(x)} + \|u\|^{\beta^-}_{1,p(x)} + \|u\|^{\beta^+}_{1,p(x)}) + 1$$

This implies that ψ is bounded on $W_0^{1,p(x)}(\Omega)$.
To show that ψ is continuous, let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Then $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(x)}(\Omega)$ and g in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \text{ and } \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq h(x) \text{ and } |\nabla u_k(x)| \leq |g(x)|$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain that

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

It follows from (f_2) that

$$|f(x, u_k(x), \nabla u_k(x))| \leq c(k(x) + |h(x)|^{r(x)-1} + |g(x)|^{r(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$k + |h|^{r(x)-1} + |g(x)|^{r(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\psi u_k - \psi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the dominated convergence theorem and the equivalence (1.1.4) imply that

$$\psi u_k \to \psi u \text{ in } L^{p'(x)}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$, and then ψ is continuous.

Step 3

Since the embedding $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is also compact. Therefore, the compositions $I^* o\phi$ and $I^* o\psi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are compact. We conclude that $S = I^* o\phi + I^* o\psi$ is compact. This completes the proof. □

Theorem 2.3.1 *Under assumptions (f_1) and (f_2), problem (2.1.1) has a weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. Let $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be as in Lemma 2.3.1 and $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, as in subsection 3.2, given by

$$\langle Lu, v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (2.1.1) if and only if

$$Lu = -Su \tag{2.3.1}$$

Thanks to the properties of the operator L seen in Theorem 2.2.1 and in view of Minty-Browder Theorem (see [87], Theorem 26A), the inverse operator $T := L^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 2.3.1 that the operator S is bounded, continuous and quasimonotone.

Consequently, equation (2.3.1) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0. \tag{2.3.2}$$

Following the terminology of [87], the equation $v + SoTv = 0$ is an *abstract Hammerstein equation* in the reflexive Banach space $W^{-1,p'(x)}(\Omega)$.

To solve equations (2.3.2), we will apply the Berkovits degree theory (1.5.3). To do this, we first claim that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega) | v + tSoTv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $\|Tv\|_{1,p(x)} = \|\nabla u\|_{p(x)}$.

If $\|\nabla u\|_{p(x)} \leq 1$, then $\|Tv\|_{1,p(x)}$ is bounded.

If $\|\nabla u\|_{p(x)} > 1$, then we get by the implication (1.1.2), the growth condition (f_2), the Hölder inequality (1.1.1), the inequality (1.1.6)and the Young inequality the estimate

$$\begin{aligned}
\|Tv\|_{1,p(x)}^{p^-} &= \|\nabla u\|_{p(x)}^{p-} \\
&\leq \rho_{p(x)}(\nabla u) \\
&= \langle Lu, u\rangle \\
&= \langle v, Tv\rangle \\
&= -t\langle SoTv, Tv\rangle \\
&= t\int_\Omega (\lambda |u|^{q(x)-2}u + f(x,u,\nabla u))u dx \\
&\leq const(|\lambda|\rho_{q(x)}(u) + \int_\Omega |k(x)u(x)|dx + \rho_{r(x)}(u) + \int_\Omega |\nabla u|^{r(x)-1}|u|dx) \\
&\leq const(\|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + \|k\|_{p'(x)}\|u\|_{p(x)} + \|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+} + \frac{1}{r'^-}\rho_{r(x)}(\nabla u) + \frac{1}{r^-}\rho_{r(x)}(u)) \\
&\leq const(\|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + \|u\|_{p(x)} + \|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+} + \|\nabla u\|_{r(x)}^{r^+}).
\end{aligned}$$

From the Poincaré inequality (1.1.8) and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$ and $L^{p(x)} \hookrightarrow L^{r(x)}$, we can deduct the estimate

$$\|Tv\|_{1,p(x)}^{p^-} \leq const(\|Tv\|_{1,p(x)}^{q^+} + \|Tv\|_{1,p(x)} + \|Tv\|_{1,p(x)}^{r^+}).$$

It follows that $\{Tv|v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (2.3.2) that the set B is bounded in $W^{-1,p'(x)}(\Omega)$. Consequently, there exists $R > 0$ such that

$$\|v\|_{-1,p'(x)} < R \text{ for all } v \in B.$$

This says that

$$v + tSoTv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1].$$

From Lemma (1.5.1) it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = LoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + SoT$ is also bounded. We conclude that

$$I + SoT \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$H(t,v) := v + tSoTv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem(1.5.3), we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

We conclude that $u = Tv$ is a weak solution of (2.1.1). This completes the proof. □

Chapitre 3

Nonlinear elliptic problems in weighted variable exponent Sobolev spaces by Topological degree

Abstract

In this chapter, we prove the existence of solutions for the nonlinear $p(\cdot)$–degenerate problems involving nonlinear operators of the form $-\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u)$ where a and f are Carathéodory functions satisfying some nonstandard growth conditions.

3.1 Introduction

Spaces with variable exponent are relevant in the study of non-Newtonian fluids. The underlying integral energy appearing in the modeling of the so called electrorheological fluids (see for instance [77, 78]) is $\int_\Omega |\nabla u|^{p(x)}dx$ or $\int_\Omega \rho(x)|\nabla u|^{p(x)}dx$. Such energies occur also in elasticity [89].

Accordingly, this naturally leads to study these fluids in the weighted variable exponent Sobolev space $W_0^{1,p(x)}(\Omega, \rho)$.

Let Ω is a bounded open subset of $I\!R^N$, $N \geq 2$. In this paper we study the problem of existence of solutions of the following nonlinear degenerated $p(x)$ elliptic problem

$$\begin{cases} -\text{ div } a(x,\nabla u) = f(x,u,\nabla u) \quad in \quad \Omega \\ u = 0 \qquad on \quad \partial\Omega. \end{cases} \tag{3.1.1}$$

where $p : \overline{\Omega} \to I\!R^+$ is a continuous function satisfying (1.2.1) and the log-Hölder continuity condition (1.1.7).

The operator $Au \equiv -\text{ div } a(x,\nabla u)$ is a Leray-Lions operator defined on $W_0^{1,p(x)}(\Omega,\rho)$, where ρ is a weight function, satisfying some integrability conditions and f is Carathéodory functions satisfying some nonstandard growth condition.

Our goal in this chapter is to prove, by using the theory of topological degree, the existence of a least weak solution of problem (3.1.1). We then extend both a class of problems involving Leray-Lions type operators with variable exponents (see [19]) and a class of some degenerate problems involving special weights [19, 30, 48].

This chapter is divided into three sections, organized as follows : in section 2, we introduce some assumptions and technical lemmas. In section 3, we prove the existence of solutions of (3.1.1).

3.2 Basic assumptions and technical lemmas

Throughout the paper, we assume that the following assumptions hold. Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, $p \in C^+(\overline{\Omega})$ and $1/p(x) + 1/p'(x) = 1$. And let ρ a weight function in Ω which satisfies (H_1), (H_2).

The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions : For all $\xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$|a(x,\xi)| \leq \beta \rho^{\frac{1}{p(x)}}(k(x) + \rho^{\frac{1}{p'(x)}}|\xi|^{p(x)-1}), \tag{3.2.1}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta, \tag{3.2.2}$$

$$a(x,\xi)\xi \geq \alpha\rho(x)|\xi|^{p(x)}, \tag{3.2.3}$$

where $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$ and α and β are a positive constants.

Let $f(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|f(x,s,\xi)| \leq \gamma \rho^{\frac{1}{p(x)}}(h(x) + \rho^{\frac{1}{p'(x)}}|s|^{q(x)-1} + \rho^{\frac{1}{p'(x)}}|\xi|^{q(x)-1}) \tag{3.2.4}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where γ is a positive constant, h is a positive function in $L^{p'(x)}(\Omega)$ and $1 < q^- \leq q(x) \leq q^+ < p^-$.

Lemma 3.2.1 *[65] Let $g \in L^{r(x)}(\Omega,\rho)$ and $g_n \in L^{r(x)}(\Omega,\rho)$ with $\|g_n\|_{r(x),\rho} \le C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e on Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega,\rho)$.*

Let define the operator $A : W_0^{1,p(x)}(\Omega,\rho) \to W^{-1,p'(x)}(\Omega,\rho^\star)$ by

$$Au = -diva(x,\nabla u) \tag{3.2.5}$$

then,

$$\langle Au, v\rangle = \int_\Omega a(x,\nabla u)\nabla v dx,$$

for all v in $W_0^{1,p(x)}(\Omega,\rho)$.

Lemma 3.2.2 *Under assumptions* (3.2.1)–(3.2.3)*, A is bounded, continuous, coercive and satisfies condition (S_+).*

Proof. **Step 1 :**

Let's show that the operator A is bounded. Firstly, by using (H_1), (3.2.1) and proposition 1.2.2 we can easily prove that $\|a(x,\nabla u)\rho(x)^{\frac{-1}{p(x)}}\|_{p'(x)}$ is bounded for all $u \in W_0^{1,p(x)}(\Omega,\rho)$. Therefore, thanks to Hölder's inequality, we have for all $u, v \in W_0^{1,p(x)}(\Omega,\rho)$,

$$\begin{aligned}
\big|\langle Au, v\rangle\big| &= \big|\int_\Omega a(x,\nabla u)\nabla v dx\big| \\
&\le C\|a(x,\nabla u)\rho(x)^{\frac{-1}{p(x)}}\|_{p'(x)}.\|\nabla v\rho(x)^{\frac{1}{p(x)}}\|_{p(x)} \\
&\le Cte\|\nabla v\rho(x)^{\frac{1}{p(x)}}\|_{p(x)} \\
&= C\|v\|_{1,p(x),\rho},
\end{aligned}$$

which implies that the operator A is bounded.

Step 2 :

To show that A is continuous, let $u_n \to u$ in $W_0^{1,p(x)}(\Omega,\rho)$. Then $\nabla u_n \to \nabla u$ in $L^{p(x)}(\Omega,\rho)$. Hence there exist a subsequence (u_k) of (u_n) and measurable function g in $(L^{p(x)}(\Omega,\rho))^N$ such that

$$\nabla u_k(x) \to \nabla u(x) \text{ and } |\nabla u_k(x)| \le g(x)$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since a satisfies the Carathéodory condition, we obtain that

$$a(x,\nabla u_k(x)) \to a(x,\nabla u(x)) \text{ a.e. } x \in \Omega.$$

it follows from 3.2.1 that

$$|a(x,\nabla u_k(x))| \le \beta\rho^{\frac{1}{p(x)}}(k(x) + \rho^{\frac{1}{p'(x)}}|g(x)|^{p(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. We have

$$\int_\Omega [\rho^{\frac{1}{p(x)}}(k(x)+\rho^{\frac{1}{p'(x)}}|g(x)|^{p(x)-1})]^{p'(x)}\rho^*(x)dx \leq C\int_\Omega (|k(x)|^{p'(x)}+\rho|g(x)|^{p(x)}) < \infty,$$

because $k \in L^{p'(x)}(\Omega)$ and $g \in (L^{p(x)}(\Omega,\rho))^N$.

Then

$$x \longmapsto \beta\rho^{\frac{1}{p(x)}}(k(x)+\rho^{\frac{1}{p'(x)}}|g(x)|^{p(x)-1}) \in L^{p'(x)}(\Omega,\rho^*),$$

and taking into account the equality

$$I_{p',\rho^*}(a(x,\nabla u_k)-a(x,\nabla u)) = \int_\Omega |a(x,\nabla u_k(x))-a(x,\nabla u(x))|^{p'(x)}\rho^*(x)dx,$$

the dominated convergence theorem and the equivalence (3) of 1.2.2 implies that

$$a(x,\nabla u_k) \to a(x,\nabla u) \text{ in } L^{p'(x)}(\Omega,\rho^*).$$

Thus the entire sequence $a(x,\nabla u_n)$ converges to $a(x,\nabla u)$ in $L^{p'(x)}(\Omega,\rho^*)$.

Then, $\forall v \in W_0^{1,p(x)}(\Omega,\rho)$; $\langle Au_n, v\rangle \to \langle Au, v\rangle$, which implies that the operator A is continuous.

Step 3

In this step, we show that A is coercive. For that Let $v \in W_0^{1,p(x)}(\Omega,\rho)$, from (3.2.3), we have by using the proposition (1.2.2) and remark (1.2.1)

$$\frac{< Av, v >}{||v||_{1,p(x),\rho}} \geq \frac{C}{||v||_{1,p(x),\rho}}\alpha I_{p,\rho}(\nabla v) \geq C'||v||^r_{1,p(x),\rho}$$

for some $r > 0$. By consequent

$$\frac{< Av, v >}{||v||_{1,p(x),\rho}} \longrightarrow \infty \quad \text{as } ||v||_{1,p(x),\rho} \to \infty.$$

Step 4

It remains to show that the operator A satisfies condition (S_+). Let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega,\rho)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega,\rho) \\ \limsup_{k\to\infty}\langle Au_n, u_n-u\rangle \leq 0. \end{cases} \tag{3.2.6}$$

We will prove that

$$u_n \longrightarrow u \text{ in } W_0^{1,p(x)}(\Omega,\rho).$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega,\rho)$, then $(u_n)_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega,\rho)$. By using the proposition 1.2.2 there is a subsequence still denoted by $(u_n)_n$ such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega,\rho),$$

$$u_n \to u \text{ in } L^{p(x)}(\Omega) \text{ and a.e in } \Omega.$$

From 3.2.2 and 3.2.6, we have

$$\lim_{n\to\infty} \langle Au_n, u_n - u\rangle = \lim_{n\to\infty} \langle Au_n - Au, u_n - u\rangle = 0. \tag{3.2.7}$$

Let $D_n = a(x, \nabla u_n).\nabla(u_n - u)$. By (3.2.7) $D_n \to 0$ in $L^1(\Omega)$ and for a subsequence $D_n \to 0$ a.e. in Ω. Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega, \rho)$ and a.e in Ω, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$. Defining $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$, we have

$$\begin{aligned} D_n(x) &= a(x, \xi_n).(\xi_n - \xi) \\ &= a(x, \xi_n)\xi_n - a(x, \xi_n)\xi \\ &\geq \alpha\rho(x)|\xi_n|^{p(x)} - \beta\rho(x)^{\frac{1}{p(x)}}(k(x) + \rho(x)^{\frac{1}{p'(x)}}|\xi_n|^{p(x)-1})|\xi| \\ &\geq \alpha|\xi_n|^{p(x)} - C_x\big[1 + |\xi_n|^{p(x)-1}\big], \end{aligned}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n, we deduce that

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|}\big).$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction. Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by the continuity of a we obtain

$$a(x, \xi^*).(\xi^* - \xi) = 0.$$

In view of (3.2.2), we have $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e.in } \Omega.$$

Since the sequence $a(x, \nabla u_n)$ is bounded in $(L^{p'(x)}(\Omega, \rho^*))^N$, and
$a(x, \nabla u_n) \to a(x, \nabla u)$ a.e. in Ω, then by Lemma 3.2.1 we get

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \quad \text{in } (L^{p'(x)}(\Omega, \rho^*))^N \text{ a.e. in } \Omega.$$

We set $\bar{y}_n = a(x, \nabla u_n)\nabla u_n$ and $\bar{y} = a(x, \nabla u)\nabla u$. We can write

$$\bar{y}_n \to \bar{y} \quad \text{in} \quad L^1(\Omega).$$

We have

$$a(x, \nabla u_n).\nabla u_n \geq \alpha\rho(x)|\nabla u_n|^{p(x)}.$$

Let $z_n = \rho|\nabla u_n|^{p(.)}$, $z = \rho|\nabla u|^{p(.)}$, $y_n = \frac{\bar{y}_n}{\alpha}$, and $y = \frac{\bar{y}}{\alpha}$. By Fatou's lemma,

$$\int_\Omega 2y\, dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\, dx;$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z| dx$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z| dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z| dx \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(\Omega, \rho))^N.$$

Hence $u_n \to u$ in $W_0^{1,p(x)}(\Omega, \rho)$, which completes the proof. □

Lemma 3.2.3 *Under assumption* (3.2.4)*, the operator*
$S : W_0^{1,p(x)}(\Omega, \rho) \to W^{-1,p'(x)}(\Omega, \rho^*)$ *setting by*

$$\langle Su, v\rangle = -\int_\Omega f(x, u, \nabla u) v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega, \rho)$$

is compact.

Proof. Let $\phi : W_0^{1,p(x)}(\Omega, \rho) \to L^{p'(x)}(\Omega, \rho^*)$ be an operator defined by

$$\phi u(x) := -f(x, u, \nabla u) \text{ for } u \in W_0^{1,p(x)}(\Omega, \rho) \text{ and } x \in \Omega.$$

By a similar proof to that of Lemma 3.2.2, we can show that ϕ is bounded and continuous.
Since the embedding $I : W_0^{1,p(x)}(\Omega, \rho) \to L^{p(x)}(\Omega, \rho)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega, \rho^*) \to W^{-1,p'(x)}(\Omega, \rho^*)$ is also compact. Therefore, the composition
$I^* o\phi : W_0^{1,p(x)}(\Omega, \rho) \to W^{-1,p'(x)}(\Omega, \rho^*)$ is compact. This completes the proof. □

3.3 Existence of weak solution

Let us first define the weak solution of problem (3.1.1).

Definition 3.3.1 A weak solution of the problem (3.1.1) is a measurable function $u \in W_0^{1,p(x)}(\Omega, \rho)$ such that

$$\int_\Omega a(x, \nabla u)\nabla v dx = \int_\Omega f(x, u, \nabla u) v dx$$

for all $v \in W_0^{1,p(x)}(\Omega, \rho)$.

The main result of this chapter is the following theorem.

Theorem 3.3.1 *Under assumptions* (3.2.1)– (3.2.4)*, there exists at least a weak solution of* (3.1.1).

Proof. Let A and $S : W_0^{1,p(x)}(\Omega,\rho) \to W^{-1,p'(x)}(\Omega,\rho^*)$ be as in (3.2.5) and Lemma 3.2.3. Then $u \in W_0^{1,p(x)}(\Omega,\rho)$ is a weak solution of (3.1.1) if and only if

$$Au = -Su \tag{3.3.1}$$

Since the operator A is strictly monotone (by assumbtion (3.2.2), thanks to the properties seen in Lemma 3.2.2 and in view of Minty-Browder Theorem (see [87], Theorem 26A), the inverse operator $T := A^{-1} : W^{-1,p'(x)}(\Omega,\rho^*) \to W_0^{1,p(x)}(\Omega,\rho)$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 3.2.3 that the operator S is bounded, continuous and quasimonotone. Consequently, equation (3.3.1) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0. \tag{3.3.2}$$

To solve equation (3.3.2), we will apply the degree theory introducing in section 2. To do this, we first claim that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega,\rho^*) | v + tSoTv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $\|Tv\|_{1,p(x),\rho} = \|\nabla u\|_{p(x),\rho}$.
If $\|\nabla u\|_{p(x),\rho} \leq 1$, then $\|Tv\|_{1,p(x),\rho}$ is bounded.
If $\|\nabla u\|_{p(x),\rho} > 1$, then we get by the proposition 1.2.2, the assumption (3.2.3) and the growth condition (3.2.4) the estimate

$$\begin{aligned}
\|Tv\|_{1,p(x),\rho}^{p^-} &= \|\nabla u\|_{p(x),\rho}^{p-} \\
&\leq I_{p,\rho}(\nabla u) \\
&\leq \frac{1}{\alpha}\langle Au, u\rangle \\
&= \frac{1}{\alpha}\langle v, Tv\rangle \\
&= \frac{-t}{\alpha}\langle SoTv, Tv\rangle \\
&= \frac{t}{\alpha}\int_\Omega f(x,u,\nabla u)u dx \\
&\leq const(\int_\Omega |\rho^{\frac{1}{p(x)}} h(x)u(x)|dx + I_{q,\rho}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|\rho(x)dx).
\end{aligned}$$

Since $h \in L^{p'(x)}(\Omega)$, then $\rho^{\frac{1}{p(\cdot)}} h \in L^{p'(x)}(\Omega,\rho^*)$. By the Hölder inequality, we have

$$\int_\Omega |\rho^{\frac{1}{p(x)}} h(x)u(x)|dx \leq 2\|\rho^{\frac{1}{p(\cdot)}} h\|_{p'(x),\rho^*}\|u\|_{p(x),\rho} = 2\|h\|_{p'(x)}\|u\|_{p(x),\rho}.$$

By the Young inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u|^{q(x)-1}|u|\rho(x)dx &= \int_\Omega |\nabla u|^{q(x)-1}\rho^{\frac{1}{q'(x)}}.|u|\rho^{\frac{1}{q(x)}}dx\\ &\leq \int_\Omega \frac{1}{q'(x)}|\nabla u|^{q(x)}\rho(x)dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\rho(x)dx\\ &= \frac{1}{q'^-}I_{q,\rho}(\nabla u) + \frac{1}{q^-}I_{q,\rho}(u)\\ &\leq const(\|\nabla u\|^{q^+}_{q(x),\rho} + \|u\|^{q^+}_{q(x),\rho} + \|u\|^{q^-}_{q(x),\rho}).\end{aligned}$$

We can then deduce

$$\|Tv\|^{p^-}_{1,p(x),\rho} \leq const(\|u\|_{p(x),\rho} + \|u\|^{q^+}_{q(x),\rho} + \|u\|^{q^-}_{q(x),\rho} + \|\nabla u\|^{q^+}_{q(x),\rho}).$$

From the Poincaré inequality and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduct the estimate

$$\|Tv\|^{p^-}_{1,p(x),\rho} \leq const(\|Tv\|_{1,p(x),\rho} + \|Tv\|^{q^+}_{1,p(x),\rho}).$$

It follows that $\{Tv|v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (3.3.2) that the set B is bounded in $W^{-1,p'(x)}(\Omega, \rho^*)$. Consequently, there exists $R > 0$ such that

$$\|v\|_{-1,p'(x),\rho^*} < R \quad \text{for all} \quad v \in B.$$

This says that

$$v + tSoTv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

From Lemma (1.5.1) it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = AoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega, \rho^*)$ given by

$$H(t,v) := v + tSoTv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem(1.5.3), we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

We conclude that $u = Tv$ is a weak solution of (3.1.1). This completes the proof. □

Chapitre 4

Nonlinear Elliptic Equations by Topological Degree in Musielak-Orlicz-Sobolev Spaces

Abstract

We prove by using the topological degree theory the existence of at least one weak solution for the nonlinear elliptic equation

$$-\operatorname{div} a_1(x,\nabla u)+a_0(x,u)=f(x,u,\nabla u)$$

with homogeneous Dirichlet boundary condition in Musielak-Orlicz-Sobolev spaces.

4.1 Introduction

Recently, there has been an increasing interest in the study of elliptic and parabolic mathematical problems in Musielak-Orlicz-Sobolev spaces. This setting includes and generalizes variable exponent, anisotropic and classical Orlicz settings.

The interest brought to the study of such differential equations comes for example from applications to non-Newtonian fluids (see [53, 54] for a wide expository) and other physics phenomena. We refer to some results on existence of solutions for Leray-Lions problems studied in variable exponent Sobolev spaces (see, e.g.,[20, 67, 85]) or Orlicz-Sobolev spaces (see, e.g., [21, 52]).

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. let us suppose that the boundary of Ω denoted $\partial\Omega$ is $\mathcal{C}^1$.

We consider a class of nonlinear Dirichlet problems of the form :

$$\begin{cases} -\text{ div } a_1(x,\nabla u) + a_0(x,u) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

The right-hand side f is a Carathéodory function which depend on the solution u and on its gradient ∇u satisfying a growth condition and where

$a_1 : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying Leray-Lions-like conditions which generate an operator of the monotone type $-$ div $a_1(x, \nabla u) + a_0(x, u)$ defined on $W_0^1 L_\Phi(\Omega)$ with values in its dual $(W_0^1 L_\Phi(\Omega))'$. Here Φ is a Musielak-Orlicz function satisfying Some sufficient conditions, namely Δ_2-condition which assure the reflexivity of such spaces.

The authors in [44] studied the problem (4.1.1) and proved the existence of weak solutions by using a linear functional analysis and sub-supersolution methods. In the case when $a_0 = 0$, the authors in [69] obtained the existence of weak solutions for (4.1.1). Bisedes, for $a_0 \neq 0$ verifying suitable conditions, the author in [51] proved the existence of weak solution with homogeneous Neumann or Dirichlet boundary condition by a sub-supersolution method.

The aim of this chapter is to prove again an existence result [44] of at least one weak solution of (4.1.1) by using a different approach opening new perspectives : we apply the degree theory in [23, 62] to give a result about existence of nonzero solutions of operator equations of the abstract Hammerstein equation in reflexive Banach spaces X

$$u + STu = 0, \quad u \in X,$$

where $S : X' \to X$ and $S : X \to X'$ two mappings [23, 62].

The approach considered here require the reflexivity of the spaces. For that, we suppose that the Musielak-Orlicz functions satisfy suitable conditions (see condition (E) below). The principal prototype that we have in mind is the Φ-Laplacien equation, i.e.

$$\text{-div}\Big(\frac{a(x,\nabla u)}{|\nabla u|}\cdot\nabla u\Big) = f(x,u,\nabla u)$$

The plan of chapter is as follows : in section 4.2, we define our basic assumptions and and give some technical lemmas . In section 4.3, we give the main result and its proof.

4.2 Basic assumptions and technical lemmas

Let Φ and Ψ are two Musielak-Orlicz functions defined on $\Omega \times \mathbb{R}^+$.

We say that Φ and Ψ satisfy the condition (E) if :

– Φ, Ψ, $\overline{\Phi}$ and $\overline{\Psi}$ are locally integrable, uniformly convex and satisfy Δ_2-condition,

– Φ satisfy the condition (1.3.1),

– $\Phi \preceq \Psi$ and the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact,

– Φ satisfies the following coerciveness condition :

there is a function ζ defined on $(0; +\infty)$ such that $\lim\limits_{s\to+\infty} \zeta(s) = +\infty$ and $\Phi(x, ts) \geq \zeta(s)s\Phi(x, t)$ for $x \in \Omega, s > 0$ and $t \in \mathbb{R}^+$.

– there is a constant c_1 such that $\inf\limits_{x\in\Omega} \Phi(x, 1) = c_1 > 0$ and for every $t_0 > 0$ there exists $c_2 = c_2(t_0)$ such that $\inf\limits_{x\in\Omega} \dfrac{\overline{\Psi}(x, t)}{t} = c_2 > 0$ for every $t \geq t_0$.

Note that under the condition (E), the spaces $L_\Phi(\Omega), L_\Psi(\Omega), W_0^1 L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are separable reflexive Banach spaces [72]. Note also that a compact embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ exist by proposition 1.3.2 and that the coerciveness condition of Φ is assured by proposition 3.1. in [44].

Let Φ and Ψ satisfying the condition (E) and $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodory functions which satisfies the growth, the coercivity and the monotony conditions : for a.e. $x \in \Omega$, for every $\xi, \xi' \in \mathbb{R}^N$ and $t, t' \in \mathbb{R}$ there is two positive constants C an C', a nonnegative function g in $L_{\overline{\Phi}}(\Omega)$ and a nonnegative function h in $L^1(\Omega)$ such that

$$|a_1(x, \xi)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |\xi|)) + g(x), \tag{4.2.1}$$

$$a_1(x, \xi).\xi \geq C'\Phi(x, |\xi|) - h(x), \tag{4.2.2}$$

$$\big(a_1(x, \xi) - a_1(x, \xi')\big) \cdot \big(\xi - \xi'\big) > 0, \qquad \xi \neq \xi', \tag{4.2.3}$$

and

$$|a_0(x, t)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |t|)) + g(x), \tag{4.2.4}$$

$$a_0(x, t)t \geq C'\Phi(x, |t|) - h(x), \tag{4.2.5}$$

$$\big(a_0(x, t) - a_0(x, t')\big)\big(t - t'\big) > 0, \qquad t \neq t', \tag{4.2.6}$$

$f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function verifying the following growth condition : there is a function q in $L_{\overline{\Phi}}(\Omega)$,a positive constant α such that

$$|f(x, t, \xi)| \leq q(x) + \alpha\overline{\Phi}^{-1}\Phi(x, |t|) + \beta\overline{\Phi}^{-1}\Phi(x, |\xi|) \tag{4.2.7}$$

for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

The Nemytsky operator F defined by f is given by

$$F(u)(x) = f(x, u(x), \nabla u(x)), \qquad x \in \Omega.$$

Lemma 4.2.1 *Let Φ a Musielak-Orlicz function such that both Φ and $\overline{\Phi}$ satisfy the Δ_2- condition. Assume* (4.2.7). *Then $F(W_0^1L_\Phi(\Omega)) \subset L_{\overline{\Psi}}(\Omega)$ and moreover, F is continuous from $W_0^1L_\Phi(\Omega)$ into $L_{\overline{\Psi}}(\Omega)$ and maps bounded sets into bounded sets.*

Proof. Let $u \in W_0^1L_\Phi(\Omega)$. For $\lambda > max(3\alpha; 3\beta)$ we have

$$\begin{aligned}
&\int_\Omega \overline{\Phi}(x, \frac{F(u)(x)}{\lambda})\,dx \\
&= \int_\Omega \overline{\Phi}(x, \frac{f(x,u(x),\nabla u(x))}{\lambda})\,dx \\
&\leq \int_\Omega \overline{\Phi}(x, \frac{1}{\lambda}[q(x) + \alpha\overline{\Phi}^{-1}\Phi(x,|u(x)|) + \beta\overline{\Phi}^{-1}(x,\Phi(x,|\xi|)])\,dx \\
&\leq \int_\Omega \frac{1}{3}\overline{\Phi}(x, \frac{3q(x)}{\lambda}) + \frac{1}{3}\Phi(x,|u(x)|) + \frac{1}{3}\Phi(x,|\nabla u(x)|)\,dx \\
&< +\infty.
\end{aligned} \tag{4.2.8}$$

By condition (E) we have $\Phi \prec \Psi$ then $\overline{\Psi} \prec \overline{\Phi}$ and by consequent there is $\lambda' > 0$ such that

$$\int_\Omega \overline{\Psi}(x, \frac{F(u)(x)}{\lambda'})\,dx < +\infty.$$

For the continuity of F, let us consider a sequence $(u_n)_n \subset W_0^1L_\Phi(\Omega)$ such that $\|u_n - u\|_{1,\Phi} \to 0$ as $n \to +\infty$ in $W^1L_\Phi(\Omega)$(we mean by $\|.\|_{1,\Phi}$ the norm of $W_0^1L_\Phi(\Omega)$ defined as the norm-closure of $\mathcal{D}(\Omega)$). Then $\|u_n - u\|_\Phi \to 0$ and $\|\nabla u_n - \nabla u\|_\Phi \to 0$ as $n \to +\infty$. Applying Lemma 1.3.1 we can find $w \in L_\Phi(\Omega)$ and extract a subsequence of $(u_n)_n$ still denoted $(u_n)_n$ such that

$$\begin{aligned}
&|u_n(x)| \leq w(x), \quad u_n(x) \to u(x) \quad \text{a.e. in } \Omega, \\
&|\nabla u_n(x)| \leq w(x), \quad \nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega.
\end{aligned} \tag{4.2.9}$$

Since f is a Carathéodory function, we obtain that

$$f(x,u_n,\nabla u_n) \to f(x,u,\nabla u) \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty$$

therefore,

$$\overline{\Phi}(x, F(u_n)(x) - F(u)(x)) \to 0 \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty.$$

By using (4.2.7), (4.2.9) and a similar argument to that in (4.2.8), there is a positive constant such that

$$\begin{aligned}
&\int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\,dx \\
&\leq c\int_\Omega \overline{\Phi}(x,q(x)) + \Phi(x,w(x)) + \Phi(x,|u(x)|) + \Phi(x,|\nabla u(x)|)\,dx
\end{aligned}$$

The right term of this inequality belongs to $L^1(\Omega)$, then by applying Lebesgue's dominated convergence theorm it follows that

$$\lim_{n\to+\infty}\int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

which implies by the continuous embedding $L_{\overline{\Phi}} \hookrightarrow L_{\overline{\Psi}}$ that

$$\lim_{n\to+\infty}\int_\Omega \overline{\Psi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

therefore the subsequence $F(u_n)$ converges to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$ for the modular convergence. By applying proposition 1.3.1 we deduce that the sequence $F(u_n)$ converges in norm to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$. The limit $F(u)$ is independent of the subsequence, by consequent this convergence hold true for the sequence $(u_n)_n$. Thus F is continuous from $W_0^1L_\Phi(\Omega)$ into $L_{\overline{\Psi}}(\Omega)$.
The functions Ψ and $\overline{\Psi}$ satisfy Δ_2-condition, then modular boundedness is equivalent to the norm boundedness. Using arguments similar to those above, F maps bounded sets of $W_0^1L_\Phi(\Omega)$ into bounded sets of $L_{\overline{\Psi}}(\Omega)$. □

Define A_1 and $A_0 : W_0^1L_\Phi(\Omega) \to (W_0^1L_\Phi(\Omega))'$ respectively for all $u, v \in W_0^1L_\Phi(\Omega)$ by

$$\langle A_1u, v\rangle = \int_\Omega a_1(x, \nabla u)v\, dx,$$

$$\langle A_0u, v\rangle = \int_\Omega a_0(x, u)v\, dx,$$

By the same way like in the proof of Theorem 2.2. an Theorem 2.3. in [51] we can proof the following lemma

Lemma 4.2.2 *Under the assumptions* (E), (4.2.1),(4.2.2), (4.2.3), (4.2.4),(4.2.5) *and* (4.2.6) *the mapping* $A := A_1 + A_0$ *is bounded, continuous and strictly monotone homeomorphism of type* (S^+).

Lemma 4.2.3 *Suppose that the assupmtions* (E),(4.2.2) *and* (4.2.5) *hold. Then* A *is coercive, i.e.,*

$$\frac{\langle Au, u\rangle}{\|u\|_{1,\Phi}} \to +\infty \text{ as } \|u\|_{1,\Phi} \to +\infty.$$

Proof. Let $u \in W_0^1L_\Phi(\Omega)$ $(u \neq 0)$ such that Φ verify the coerciveness condition (see condition (E) below), by using (4.2.2) and (4.2.5) we have

$$\begin{aligned}
\langle Au, u\rangle &= \int_\Omega a_1(x, \nabla u)\cdot\nabla u + a_0(x, u)u\, dx\\
&\geq 2C'\Big(\int_\Omega \Phi(x, |\nabla u|) + \Phi(x, |u|) - h(x)\, dx\Big)\\
&\geq 2C'\Big(\int_\Omega \Phi(x, \frac{\|u\|_{1,\Phi}|\nabla u|}{\|u\|_{1,\Phi}}) + \Phi(x, \frac{\|u\|_{1,\Phi}|u|}{\|u\|_{1,\Phi}})\, dx\Big) - 2\|h\|_{L^1(\Omega)}
\end{aligned}$$

$$\geq 2C'\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi}(\int_\Omega \Phi(x,\frac{|\nabla u|}{\|u\|_{1,\Phi}})+\Phi(x,\frac{|u|}{\|u\|_{1,\Phi}})\,dx)-2\|h\|_{L^1(\Omega)}$$

We have $\|\frac{|\nabla u|}{\|u\|_{1,\Phi}}\|_{1,\Phi}\leq 1$, $\|\frac{|u|}{\|u\|_{1,\Phi}}\|_{1,\Phi}\leq 1$ and

$$\lim_{\|u\|_{1,\Phi}\to+\infty}\zeta(\|u\|_{1,\Phi})=+\infty,$$

therefore $\dfrac{\langle Au,u\rangle}{\|u\|_{1,\Phi}}\to+\infty$ as $\|u\|_{1,\Phi}\to+\infty$. □

By applying Minty-Browder theorem (or Lemma 4.2.3 and Lemma 5.2. in [23]), we deduce that the inverse operator $T:(W_0^1L_\Phi(\Omega))'\to W_0^1L_\Phi(\Omega)$ of A is also bounded, continuous and of type (S^+). On other hand, by the condition (E), the embedding $I:W_0^1L_\Phi(\Omega)\to L_\Psi(\Omega)$ is compact, by consequent the adjoint operator $I^*:L_{\overline{\Psi}}(\Omega)\to(W_0^1L_\Phi(\Omega))'$ is also compact. On other hand, the continuity and boundedness of Nemytsky operator F proved in Lemma 4.2.1 implies that the composition $S:=-I^*oF$ is compact. Consequently we have the following lemma

Lemma 4.2.4 *The mapping $S:W_0^1L_\Phi(\Omega)\to(W_0^1L_\Phi(\Omega))'$ is continuous and compact, in particular it is quasimonotone.*

4.3 Existence result

Let us give a definition of a weak solution of problem (4.1.1) :

Definition 4.3.1 *A function u is called weak solution for* (4.1.1) *if $u\in W_0^1L_\Phi(\Omega)$, $F(u)\in L_{\overline{\Psi}}(\Omega)$ and*

$$\int_\Omega a_1(x,\nabla u)v\,dx+\int_\Omega a_0(x,u)v\,dx=\int_\Omega f(x,u,\nabla u)v\,dx,\quad \textit{for all } v\in W_0^1L_\Phi(\Omega). \tag{4.3.1}$$

Theorem 4.3.1 *Let Φ and Ψ satisfy the condition (E). Suppose that the assumptions* (4.2.1)–(4.2.7) *hold true. Then there exists at least one weak solution of problem* (4.1.1).

Proof. The weak formulation (4.3.1) is equivalent to the abstract Hammerstein equation

$$(I+SoT)v=0,\qquad \text{and}\quad u=Tv. \tag{4.3.2}$$

where S and T are the maps defined in Lemme 4.2.2 and Lemma 4.2.4. To solve equation 4.3.2, We can proceed with degree theoretic arguments, it suffices to prove the boundedness of solution set of the homotopy equation

$$v+tSoTv=0,\quad v\in(W_0^1L_\Phi(\Omega))',\quad t\in[0,1],$$

Let

$$B = \{v \in (W_0^1 L_\Phi(\Omega))'; v + tSoTv = 0, \quad v \in X, \quad \text{for some } t \in [0,1]\}$$

let $v \in B$ and $u \in W_0^1 L_\Phi(\Omega)$ such that $Tv = u$, we have for some t in $[0,1]$

$$\begin{aligned}
\langle v, Tv\rangle &= \langle Au, u\rangle \\
&= -t\langle SoTv, Tv\rangle \\
&= t\int_\Omega f(x,u,\nabla u)u\,dx \\
&\leq \int_\Omega |f(x,u,\nabla u)||u|\,dx.
\end{aligned}$$

As in the proof of Lemma 4.2.3, there two positive constants C and $\widetilde{C}$ such that

$$\langle Au, u\rangle \geq C\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi} - \widetilde{C} \tag{4.3.3}$$

Let $\lambda > max(3\alpha; 3\beta)$. Since Φ satisfy the Δ_2-condition, then by using proposition 2.3 in [44], there is a function $\gamma \in L^1(\Omega)$ and a constant c such that

$$\Phi(x, \lambda|u(x)|) \leq c\Phi(x, |u(x)|) + \gamma(x)$$

which implies, by using the young's inequality, that

$$\begin{aligned}
&|f(x,u,\nabla u)||u| \\
&\leq \overline{\Phi}(x, \frac{|f(x,u,\nabla u)|}{\lambda}) + \Phi(x, \lambda|u|) \\
&\leq \frac{1}{3}\overline{\Phi}(x, \frac{3q(x)}{\lambda}) + \frac{1}{3}\Phi(x,|u(x)|) + \frac{1}{3}\Phi(x,|\nabla u(x)|) + c\Phi(x,|u(x)|) + \gamma(x).
\end{aligned} \tag{4.3.4}$$

by combining (4.3.3) and (4.3.4) we can find two constants C' and $\widetilde{C'}$ such that

$$\|u\|_{1,\Phi}(\zeta(\|u\|_{1,\Phi}) - C') \leq \widetilde{C'}$$

which implies that $u = Tv$ remain bounded in $W_0^1 L_\Phi(\Omega)$, consequently, there exists $R > 0$ such that

$$\|v\|_{(W_0^1 L_\Phi(\Omega))'} \leq R \quad \forall v \in B.$$

We deduce that for all $t \in [0,1]$,

$$v + tSoTv \neq 0, \quad \forall v \in \partial B_R(0).$$

According to Lemma 1.5.1, the Hammersein operator $I + SoT$ belongs to the class $\mathcal{F}_T(\overline{B_R(0)})$.
Let us consider the homotopy $\mathcal{H} : [0,1] \times \overline{B_R(0)} \to (W_0^1 L_\Phi(\Omega))'$ defined by

$$\mathcal{H}(t,v) = v + tSoTv.$$

By invariance and normalisation properties of the degree d of the class $\mathcal{F}_T$ (see Theorem 1.5.3) we deduce that

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

By Theorem 1.5.3 we conclude that there is at least one $\overline{v} \in B_R(0)$ verifying

$$\overline{v} + SoT\overline{v} = 0.$$

Thus $\overline{u} = T\overline{v}$ is a weak solution of problem (4.1.1). □

Deuxième partie

Construction and application of a Topological Degree theory using a complementary system formed of Generalized Sobolev Spaces

Chapitre 5

Construction of a Topological Degree theory in Generalized Sobolev Spaces

Abstract

In this chapter, we construct an integer-valued degree function in a suitable classes of mappings of monotone type, using a complementary system formed of Generalized Sobolev Spaces in which the variable exponent $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < \infty$, where $\Omega \subset \mathbb{R}^N$ is open and bounded. This kind of spaces are not necessarily reflexive.

5.1 Introduction

Topological degree theory is one of the most effective tools in solving nonlinear equations.

Brouwer had published a degree theory in 1912 for continuous maps defined in finite dimensional Euclidean space [31]. Leray and Schauder developed the degree theory for compact operators in infinite dimensional Banach spaces [68]. Since then numerous generalizations and applications have been investigated in various ways of approach (see e.g.[42, 82, 83, 86]). Browder introduced a topological degree for nonlinear operators of monotone type in reflexive Banach spaces [32, 35]. The theory was constructed later by Berkovits and Mustonen [23, 25, 26] by using the Leray-Schauder degree which can be applied to partial differential operators of general divergence form.

The purpose of this chapter is to generalize this theory to Sobolev spaces with variable exponent in the case where these spaces are not necessarily reflexive, exactly in the case where the variable exponent p satisfies $1 \leq p^- \leq p^+ < \infty$. We will construct this theory for appropriate classes of monotone mappings using a complementary system formed of Generalized Sobolev spaces.

The chapter is divided into three parts. In the second section, we introduce some preliminary definitions and results concerning the generalized Lebesgue and Sobolev spaces, we construct a complementary system of these spaces. The third section is dedicated to the construction of degree theory in generalized Sobolev spaces.

5.2 Preliminary definitions and results

In the sequel, we consider a naturel number $N \geq 1$ and an open and bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$.

5.2.1 Generalized Lebesgue spaces

We define $\mathcal{P}(\Omega)$ to be the set of all measurable function : $p : \Omega \to [1, +\infty]$. Functions $p \in \mathcal{P}(\Omega)$ are called variable exponents on Ω. We define $p^- = \operatorname{ess}\inf_{\Omega} p$ and $p^+ = \operatorname{ess}\sup_{\Omega} p$.
If $p \in \mathcal{P}(\Omega)$, then we define $p' \in \mathcal{P}(\Omega)$ by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, where $\frac{1}{\infty} := 0$. The function p' is called the dual variable exponent of p.
We say that a function $\alpha : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous* on Ω if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. We say that α satisfies the *log-Hölder decay* condition if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{log(e + |x|)}$$

for all $x \in \Omega$. We say that α is *globally log-Hölder continuous* in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.
We define the following class of variable exponents

$$\mathcal{P}^{log}(\Omega) := \{p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous } \}.$$

We can deduce that $p \in \mathcal{P}^{log}(\Omega)$ if and only if $p' \in \mathcal{P}^{log}(\Omega)$.

For $t \geq 0$, $x \in \Omega$ and $1 \leq p < \infty$ we define

$$\varphi_{p(x)}(t) := t^{p(x)}$$

Moreover we set

$$\varphi_\infty(t) := \infty.\chi_{(1,\infty)}(t) = \begin{cases} 0 & \text{if } t \in [0,1] \quad , \\ \infty & \text{if } t \in (1,\infty) \quad . \end{cases}$$

We will use $t^{p(x)}$ as an abbreviation for $\varphi_{p(x)}(t)$, also in the case $p = \infty$. Similarly, $t^{\frac{1}{p(x)}}$ will denote the inverse function $\varphi_{p(x)}^{-1}(t)$; note that in case $p = \infty$ we have $t^{\frac{1}{\infty}} = \varphi_\infty^{-1}(t) = \chi_{(0,\infty)}(t)$.

For any variable exponent $p(\cdot)$ and any measurable function u, we define the modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx,$$

and we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{p(\cdot)}(\lambda u) < \infty \text{ for some} \lambda > 0\}$$

equipped with the norm, called the Luxemburg norm,

$$||u||_{p(\cdot)} = \inf\{\lambda > 0 / \rho_{p(.)}(\frac{u}{\lambda}) \leq 1\}.$$

It is a Banach space ([43, Theorem 3.2.7]). The space $E^{p(\cdot)}(\Omega)$ is the closure of the space $L^\infty(\Omega)$ with respect to the Luxemburg norm.

Theorem 5.2.1 *[43] Let $p(\cdot)$ and $q(\cdot)$ be the exponent and $\Omega \subset \mathbb{R}^N$ open and bounded. Then*

(i) $E^{p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$,

(ii) $E^{p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$ *iff* $p^+ < \infty$,

(iii) $E^{p(\cdot)}(\Omega)$ *is separable,*

(iv) $(E^{p(\cdot)}(\Omega))^* = L^{p'(\cdot)}(\Omega)$,

(v) $L^{p(\cdot)}(\Omega)$ *is reflexive iff* $1 < p^- \leq p(x) \leq p^+ < \infty$.

We say that a sequence $\{u_n\} \subset L^{p(\cdot)}(\Omega)$ converges to $u \in L^{p(\cdot)}(\Omega)$ *in the modular sense,* denote $u_n \to u(mod)$ in $L^{p(\cdot)}$, if there exists $\lambda > 0$ such that

$$\rho_{p(\cdot)}(\frac{u_n - u}{\lambda}) \to 0, \text{ when } n \to \infty.$$

Let X and Y be arbitrary Banach spaces with bilinear bicontinuous pairing $\langle .,. \rangle_{X,Y}$. We say that a sequence $u_n \subset X$ converges to $u \in X$ with respect the topology $\sigma(X,Y)$, denote $u_n \to u(\sigma(X,Y))$ in X, if $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in Y$. When $Y^* \cong X$, we denote only $u_n \rightharpoonup u$ in X.

Theorem 5.2.2 *([43, 63]) In any Generalized Lebesgue space $L^{p(\cdot)}(\Omega)$*

(i) *norm convergence implies modular convergence,*

(ii) *norm convergence and modular convergence are equivalent iff* $p^+ < \infty$,

(iii) *modular convergence implies* $\sigma(L^{p(\cdot)}, L^{p'(\cdot)})$ *convergence.*

5.2.2 Complementary system of Generalized Sobolev spaces

Definition 5.2.1 *Let Y and Z be Banach spaces in duality with respect with to a continuous pairing $\langle ., . \rangle$ and let Y_0 and Z_0 be closed subspaces of Y and Z respectively. Then the quadruple $\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix}$ is caled a complementary system* if, by means of $\langle ., . \rangle$, $Y_0^* \cong Z$ and $Z_0^* \cong Y$.

An example of a complementary system is

$$\begin{pmatrix} L^{p(\cdot)}(\Omega) & L^{p'(\cdot)}(\Omega) \\ E^{p(\cdot)}(\Omega) & E^{p'(\cdot)}(\Omega) \end{pmatrix}$$

The following lemma gives an important method by which from a complementary system $\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix}$ and a closed subspace E of Y, one can construct a new complementary system $\begin{pmatrix} E & F \\ E_0 & F_0 \end{pmatrix}$. Define $E_0 = E \cap Y_0$, $F = Z/E_0^{\perp}$ and $F_0 = \{z + E_0^{\perp}; z \in Z_0\} \subset F$, where $\perp$ denotes the orthogonal in the duality (Y, Z), i.e. $E_0^{\perp} = \{z \in Z; \langle y, z \rangle = 0 \text{ for all } y \in E_0\}$.

Lemma 5.2.1 *[52, Lemma 1.2] The pairing $\langle ., . \rangle$ between Y and Z induces a pairing between E and F if and only if E_0 is $\sigma(Y, Z)$ dense in E. In this case, $\begin{pmatrix} E & F \\ E_0 & F_0 \end{pmatrix}$ is a complementary system if E is $\sigma(Y, Z_0)$ closed, and conversely, when Z_0 is complete, E is $\sigma(Y, Z_0)$ closed if $\begin{pmatrix} E & F \\ E_0 & F_0 \end{pmatrix}$ is a complementary system.*

Next, let $p \in \mathcal{P}(\Omega)$ and $m \in \mathbb{N}$.
We define the spaces

$$W^{m,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega) : D^{\alpha}u \in L^{p(.)}(\Omega), |\alpha| \leq m\},$$

$$H^{m,p(.)}(\Omega) = \{u \in E^{p(.)}(\Omega) : D^{\alpha}u \in E^{p(.)}(\Omega), |\alpha| \leq m\}$$

with the norm

$$||u||_{m,p(.)} = \sum_{|\alpha| \leq m} ||D^{\alpha}u||_{p(.)}.$$

The spaces $W^{m,p(.)}(\Omega)$ and $H^{m,p(.)}(\Omega)$ are Banach spaces (see [43]).
We say that a sequence $\{u_n\} \subset W^{m,p(.)}(\Omega)$ converges to $u \in W^{m,p(.)}(\Omega)$ *in the modular sense,* denote

$u_n \to u(mod)$ in $W^{m,p(.)}$, if there exists $\lambda > 0$ such that

$$\rho_{p(.)}(\frac{D^\alpha u_n - D^\alpha u}{\lambda}) \to 0, \text{ when } n \to \infty,$$

for $|\alpha| \leq m$.

The Sobolev space $W_0^{m,p(.)}(\Omega)$ with zero boundary valus is the closure of the set of $W^{m,p(.)}(\Omega)$-functions with compact support, i.e.

$$\{u \in W^{m,p(.)}(\Omega) : u = u\chi_K \text{ for a compact } K \subset \Omega\}$$

in $W^{m,p(.)}(\Omega)$.

The space $W^{m,p(.)}(\Omega)$ will always be identified to a subspace of the product $\Pi_{|\alpha|\leq m} L^{p(.)} = \Pi L^{p(.)}$; this subspace is $\sigma(\Pi L^{p(.)}, \Pi E^{p'(.)})$ closed and $W_0^{m,p(.)}(\Omega)$ will be the $\sigma(\Pi L^{p(.)}, \Pi E^{p'(.)})$ closure of $\mathcal{D}(\Omega) = \bigcap_{m=1}^{\infty} C_0^m(\Omega)$ in $W^{m,p(.)}(\Omega)$.

The (norm) closure of $\mathcal{D}(\Omega)$ in the space $W^{m,p(.)}(\Omega)$ (or in $\Pi L^{p(.)}$) is denoted by $H_0^{m,p(.)}(\Omega)$.

If $p \in \mathcal{P}^{log}(\Omega)$ is bounded, then $W_0^{m,p(.)}(\Omega) = H_0^{m,p(.)}(\Omega)$ [43, Corollary 11.2.4]. The space $W_0^{m,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$ [43, Theorem 8.1.13].

Let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 < p'^- \leq p'^+ \leq \infty$. We denote the dual spaces of Sobolev spaces $W_0^{m,p(.)}(\Omega)$ and $H_0^{m,p(.)}(\Omega)$ as follows

$$W^{-m,p'(.)}(\Omega) := [W_0^{m,p(.)}(\Omega)]^\star \text{ and } H^{-m,p'(.)}(\Omega) := [H_0^{m,p(.)}(\Omega)]^\star.$$

Proposition 5.2.1 *[43, Proposition 12.3.2] Let $\Omega \subset \mathbb{R}^N$ be a domain, let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 < p'^- \leq p'^+ \leq \infty$ and let $m \in \mathbb{N}$. For each $F \in W^{-m,p'(.)}(\Omega)$ there exists $f_\alpha \in L^{p'(.)}(\Omega)$, $|\alpha| \leq m$, such that*

$$\langle F, u \rangle = \sum_{|\alpha|\leq m} \int_\Omega f_\alpha D^\alpha u \, dx$$

for all $W_0^{m,p(.)}(\Omega)$. Moreover,

$$\|F\|_{-m,p'(\cdot)} \approx \sum_{|\alpha|\leq m} \|f_\alpha\|_{p'(\cdot)}.$$

We can write an analogue proposition for $H^{-m,p'(.)}(\Omega)$ and then

$$W^{-m,p'(.)}(\Omega) = \{F \in \mathcal{D}'(\Omega) : F = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \text{ where } f_\alpha \in L^{p'(.)}(\Omega)\},$$

$$H^{-m,p'(.)}(\Omega) = \{F \in \mathcal{D}'(\Omega) : F = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \text{ where } f_\alpha \in E^{p'(.)}(\Omega)\}.$$

By Lemma 5.2.1,the quadruple

$$\begin{pmatrix} W_0^{m,p(.)}(\Omega) & W^{-m,p'(.)}(\Omega) \\ H_0^{m,p(.)}(\Omega) & H^{-m,p'(.)}(\Omega) \end{pmatrix}$$

forms a complementary system.

We say that a sequence $\{u_n\} \subset W^{-m,p'(.)}(\Omega)$ converges to $u \in W^{-m,p'(.)}(\Omega)$ *in the modular sense,* denote $u_n \to u(mod)$ in $W^{-m,p'(.)}$, if u_n and u have representations

$$u_n = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha g_\alpha^{(n)},\ u = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha g_\alpha,$$

such that $g_\alpha^{(n)}, g_\alpha \in L^{p'(.)}(\Omega)$ and $g_\alpha^{(n)} \to g_\alpha(mod)$ in $L^{p'(.)}$ for all $|\alpha| \leq m.$

Let A be a subset of a Generalized Sobolev Space Y. We denote by $\bar{A}^{mod}$ the sequentiel modular closure of A,i.e.

$$\bar{A}^{mod} = \{u \in Y/ \text{ there exists } \{u_n\} \subset A \text{ such that } u_n \to u(mod) \text{ in } Y\}.$$

5.3 Degree theory in Generalized Sobolev spaces

5.3.1 Construction of a degree function in Generalized Sobolev Spaces

Let

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^{m,p(.)}(\Omega) & W^{-m,p'(.)}(\Omega) \\ H_0^{m,p(.)}(\Omega) & H^{-m,p'(.)}(\Omega) \end{pmatrix}$$

be a complementary system formed of Generalized Sobolev Spaces in which
$\Omega \subset \mathbb{R}^N$ is open and bounded with a Lipschitz boundary $\partial\Omega$, and $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < \infty$.
We define the class $\mathcal{F}$ of *admissible mappings* and the class $\mathcal{H}$ of *admissible homotopies* as follows :
$F : D_F \subset Y \to Z$ belongs to $\mathcal{F}$, if

(a) F is a strongly quasibounded mapping of the class (MOD), i.e if the conditions $\{u_n\} \subset D_F$, $u_n \rightharpoonup u$ in Y , $F(u_n) \rightharpoonup \chi$ in Z and $limsup_{n\to\infty}\langle F(u_n), u_n\rangle \leq \langle \chi, u\rangle$ imply that $u \in D_F$, $\chi = F(u)$ and there exists a subsequence $\{u_{n'}\}$ such that $u_{n'} \to u$ (mod) in Y and $F(u_{n'}) \to F(u)$ (mod) in Z. we denote $F \in (MOD)$.

$F : D_F \subset Y \to Z$ belongs to $\mathcal{F}^a$, if $F \in \mathcal{F}$ and the following conditions hold :

(b) if $\{u_n\} \subset D_F$ is bounded , $t_n \to 0^+$ and $\langle t_n F(u_n), u_n - \bar{u}\rangle$ is bounded from above for some $\bar{u} \in Y$, then $\{t_n F(u_n)\} \subset Z$ is bounded,

(c) if $\{u_n\} \subset D_F$, $u_n \to u \in Y$ for $\sigma(Y, Z_0)$,$t_n \to 0^+$, $t_nF(u_n) \to \chi \in Z$ for $\sigma(Z, Y_0)$ and $limsup\langle t_nF(u_n), u_n\rangle \leq \langle \chi, u\rangle$, then $\langle t_nF(u_n), u_n\rangle \to \langle \chi, u\rangle$,

(d) if $\{u_n\} \subset D_F$, $u_n \to u$ (mod) in Y, $t_n \to 0^+$, $t_nF(u_n) \to \chi \in Z(\sigma(Z, Y_0))$ in Z and $limsup\langle t_nF(u_n), u_n\rangle \leq \langle \chi, u\rangle$, then $t_nF(u_n) \to 0$ (mod) in Z.

The homotopy $H : D_H \to Z$ belongs to $\mathcal{H}$, if H is a strongly quasibounded homotopy of the class (MOD).

Lemma 5.3.1 *If $F, G \in \mathcal{F}^a$, then $H(t, u) = tF(u) + (1-t)G(u)$ belongs to $\mathcal{H}$ with*

$$D_{H_t} = \begin{cases} D_F \cap D_G, \text{ if } 0 < t < 1 \\ D_G, \text{ if } t = 0 \\ D_F, \text{ if } t = 1. \end{cases}$$

Proof. **Step 1**

Since F and G are finitely continuous, the homotopy H is finitely continuous from the norm topology of $[0, 1] \times Y$ to the weak topology of Z.

Step 2

We shall prove that H is strongly quasibounded.

Assume $\{t_n\} \subset [0, 1]$, $\{u_n\} \subset D_{H_{t_n}}$ and $\langle H(t_n, u_n), u_n - \bar{u}\rangle$ is bounded from above for some $\bar{u} \in Y_0$. It follows that $\langle t_{n'}F(u_{n'}), u_{n'} - \bar{u}\rangle$ or $\langle(1 - t_{n'})G(u_{n'}), u_{n'} - \bar{u}\rangle$ is bounded from above for a subsequence. We may also suppose that $\langle t_{n'}F(u_{n'}), u_{n'} - \bar{u}\rangle$ is bounded from above. By condition **(b)** and the fact that F is strongly quasibounded, the sequence $\{t_{n'}F(u_{n'})\}$ is bounded in Z. Consequently, $\langle t_{n'}F(u_{n'}), u_{n'} - \bar{u}\rangle$ is bounded implying that $\langle(1 - t_{n'})G(u_{n'}), u_{n'} - \bar{u}\rangle$ is bounded from above. Therefore $\{(1 - t_{n'})G(u_{n'})\}$ is also bounded in Z and hence $\{H(t_{n'}, u_{n'})\}$ is bounded in Z.

By contradiction argument, $\{H(t_n, u_n)\}$ is bounded in Z.

Step 3

We shall next prove that H is a homotopy of the class (MOD).

Asuume that $t_n \subset [0, 1]$, $t_n \to t \in [0, 1]$, $\{u_n\} \subset D_{H_{t_n}}$, $u_n \rightharpoonup u$ in Y, $H(t_n, u_n) \rightharpoonup \chi$ in Z and $limsup_{n\to\infty}\langle H(t_n, u_n), u_n\rangle \leq \langle \chi, u\rangle$.

Deducing as above, $\{t_{n'}F(u_{n'})$ and $\{(1 - t_{n'})G(u_{n'})\}$ are bounded in Z for some subsequence. We may assume that $t_{n'}F(u_{n'}) \rightharpoonup \chi_1$ and $(1 - t_{n'})G(u_{n'}) \rightharpoonup \chi_2$ in Z. It is clear that $\chi = \chi_1 + \chi_2$. We may assume that $limsup_{n\to\infty}\langle t_{n'}F(u_{n'}), u_{n'}\rangle \leq \langle \chi_1, u\rangle$ or $limsup_{n\to\infty}\langle(1 - t_{n'})G(u_{n'}), u_{n'}\rangle \leq \langle \chi_2, u\rangle$. Suppose, for example, that

$$limsup_{n\to\infty}\langle t_{n'}F(u_{n'}), u_{n'}\rangle \leq \langle \chi_1, u\rangle.$$

By condition $(\mathbf{c})$ and the fact that F belongs to the class (MOD), we have $\langle t_{n'}F(u_{n'}), u_{n'}\rangle \to \langle \chi_1, u\rangle$. Hence

$$limsup_{n\to\infty}\langle(1-t_{n'})G(u_{n'}), u_{n'}\rangle \le \langle\chi_2, u\rangle.$$

If $0 < t < 1$, then $F(u_{n'}) \rightharpoonup \frac{\chi_1}{t}$ in Z, $G(u_{n'}) \rightharpoonup \frac{\chi_2}{1-t}$ in Z, $\langle F(u_{n'}), u_{n'}\rangle \to \langle \frac{1}{t}\chi_1, u\rangle$ and $limsup_{n\to\infty}\langle G(u_{n'}), u_{n'}\rangle \le \langle\frac{1}{1-t}\chi_2, u\rangle$.

Therefore $u \in D_F \cap D_G$, $\frac{\chi_1}{t} = F(u)$, $\frac{\chi_2}{1-t} = G(u)$, $u_{n'} \to u(mod)$ in Y and $F(u_{n'}) \to F(u)$, $G(u_{n'}) \to G(u)(mod)$ in Z. Hence $u \in D_{H_t}$ and $H(t_{n'}, u_{n'}) \to H(t,u)(mod)$ in Z. If $t = 0$, then $G(u_{n'}) \rightharpoonup \frac{\chi_2}{1-t}$ in Z and $limsup_{n\to\infty}\langle G(u_{n'}), u_{n'}\rangle \le \langle\frac{1}{1-t}\chi_2, u\rangle$.

Therefore $u \in D_G$, $u_{n'} \to u(mod)$ in Y and $G(u_{n'} \to G(u)(mod)$ in Z. Moreover, by condition $(\mathbf{d})$, we have $t_{n'}F(u_{n'}) \to 0$ in the modular sense. Hence $u \in D_{H_0}$ and $H(t_{n'}, u_{n'}) \to H(t,u)(mod)$ in Z. If $t = 1$, we make an analogue deduction to obtain $u \in D_F = D_{H_1}$ and $u_{n'} \to u(mod)$ in Y and $F(u_{n'}) \to F(u)(mod)$ in Z.

Consequently, H is a homotopy of the class (MOD). □

Our aim in this subsection is to construct an integer-valued degree function $d(F,G,f)$ for $F \in \mathcal{F}$, $G \subset Y_0$ open and bounded in Y_0, $f \in Z_0$ and $f \notin F(\overline{\partial_{Y_0}G}^{mod}) \cap \overline{F(\partial_{Y_0}G}^{mod}$ satisfying the following conditions :

(C_1) Existence : if $d(F,G,f) \neq 0$, then $f \in F(\bar{G}^{mod}) \cap \overline{F(G)}^{mod}$,

(C_2) Additivity : if $G_1, G_2 \subset G$ are open and bounded, $G_1 \cap G_2 = \emptyset$ and
$f \notin F(\overline{\bar{G} \setminus (G_1 \cup G_2)}^{mod}) \cap \overline{F(\bar{G} \setminus (G_1 \cup G_2))}^{mod}$,
then

$$d(F,G,f) = d(F,G_1,f) + d(F,G_2,f),$$

(C_3) Homotopy invariance : if $H \in \mathcal{H}$,$f \in Z_0$ and $f \notin H([0,1] \times \overline{\partial_{Y_0}G}^{mod}) \cap \overline{H([0,1] \times \partial_{Y_0}G)}^{mod}$, then

$$d(H(t,\cdot), G, f) = constant \text{ for all } t \in [0,1],$$

(C_4) Normalization : There exists a normalising map $K \in \mathcal{F}^a$ such that if $f \in Z_0$,
$f \notin K(\overline{\partial_{Y_0}G}^{mod}) \cap \overline{K(\partial_{Y_0}G)}^{mod}$ and $f \in K(G)$, then

$$d(K,G,f) = 1.$$

Remark 5.3.1 *We shall always assume in the applications that*
$1 < p^- \le p(\cdot) \le p^+ < \infty$. *This restriction means that instad of modular closure we have norm closures. In these cases the corresponding degree theories can be formulated as fellows :*
$F \in \mathcal{F}, G \subset Y$ *open and bounded in* Y, $f \in Z_0$ *and* $f \notin F(\partial_Y G)$

(c_1) *Existence : if* $d(F,G,f) \neq 0$ *, then* $f \in F(G)$*,*

(c_2) *Additivity : if* $G_1, G_2 \subset G$ *are open and bounded,* $G_1 \cap G_2 = \emptyset$ *and* $f \notin F(\bar{G} \setminus (G_1 \cup G_2))$*, then*

$$d(F,G,f) = d(F,G_1,f) + d(F,G_2,f),$$

(c_3) *Homotopy invariance : if* $H \in \mathcal{H}, f \in Z_0$ *and* $f \notin H([0,1] \times \partial_Y G)$*, then*

$$d(H(t,\cdot),G,f) = constant \text{ for all } t \in [0,1],$$

(c_4) *Normalization : There exists a normalising map* $K \in \mathcal{F}^a$ *such that if* $f \in Z_0$*,* $f \notin K(\partial_Y G)$ *and* $f \in K(G)$*, then*

$$d(K,G,f) = 1.$$

For the construction of such degree, we need the following :

Definition 5.3.1 *Let* Λ *be the set of all finite dimensional subspaces of* Y_0*.*
Denote
$Y_0 = X_\lambda \oplus Y_\lambda$ *for all* $X_\lambda \in \Lambda$*, where* Y_λ *is the closed complement of* X_λ *([75, p.157]),*
$A_\lambda = A \cap X_\lambda$*, when* $A \subset Y$*,*
$P_\lambda : Y_0 \to X_\lambda$ *the projection map,*
$P_\lambda^* : X_\lambda \to Z$*,* $\langle P_\lambda^*(u), v\rangle = \langle u, P_\lambda(v)\rangle$ *for all* $u \in X_\lambda$ *and* $v \in Y_0$*.*
For the natural injection $\phi_\lambda : X_\lambda \to Y_0$*, we define*
$\phi_\lambda^* : Z \to X_\lambda$*,* $\langle \phi_\lambda^*(u), v\rangle = \langle u, \phi_\lambda(v)\rangle$ *for all* $u \in Z$ *and* $v \in X_\lambda$*.*
If $F : Y_0 \to Z$*, then* $F_\lambda : X_\lambda \to X_\lambda$*,* $F_\lambda(x) = \phi_\lambda^*(F(\phi_\lambda(x)))$*.*

Let d_n be the Brouwer degree for continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Lemma 5.3.2 *[32] Let* $G \subset \mathbb{R}^n$ *be open and bounded and*
$F : \bar{G} \subset \mathbb{R}^n \to \mathbb{R}^n$ *continuous such that* $0 \notin F(\partial G)$*. Define a mapping*

$$F' : \bar{G} \times [-1,1]^m \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$$

$$F'(x,y) = (F(x), y) \text{ for all } x \in \bar{G}, y \in [-1,1]^m.$$

Then

$$d_n(F,G,0) = d_{n+m}(F', G \times (-1,1)^m, 0).$$

Let d_λ be the Brouwer degree in the space X_λ.

Lemma 5.3.3 *Let $G \subset Y_0$ be open and bounded in Y_0, $X_\lambda \subset X_\mu \in \Lambda$, such that $G \cap X_\lambda \neq \emptyset$ and $F \in (MOD)$. If $d_\lambda(F_\lambda, G_\lambda, 0) \neq d_\mu(F_\mu, G_\mu, 0)$ or one of the degrees is not defined, then there exists $u \in \partial_{Y_0} G$ such that*

$$\langle F(u), u\rangle \leq 0 \text{ and } \langle F(u), v\rangle = 0 \text{ for all } v \in X_\lambda.$$

Proof. If one of the degrees $d_\lambda(F_\lambda, G_\lambda, 0)$ and $d_\mu(F_\mu, G_\mu, 0)$ is not defined, there exists $u \in \partial_{X_\lambda} G_\lambda \subset \partial_{Y_0} G$ such that $F_\lambda(u) = 0$ or $u \in \partial_{X_\mu} G_\mu \subset \partial_{Y_0} G$ such that $F_\mu(u) = 0$. In both cases the proof is complete.

Otherwise we define a continuous mapping $S : X\mu \to X_\mu$,

$$S(x) = \phi_\lambda^*(F(P_\lambda(x))) + x - P_\lambda(x.)$$

By Lemma 5.3.2 we have

$$d_\lambda(F_\lambda, G_\lambda, 0) = d_\mu(S, G_\lambda \times (-1, 1)^m, 0),$$

where $m = dimX_\mu - dimX_\lambda$. If $S(x) = 0$ for some $x \in G_\mu$, then $x - P_\lambda(x) = 0$, implying $x \in G_\lambda \times (-1, 1)^m$. If $S(x) = 0$ for some $x \in G_\lambda \times (-1, 1)^m$ then $x - P_\lambda(x) = 0$, which means that $x \in G_\lambda \subset G_\mu$. By the excision property of the Brouwer degree, we have

$$d_\mu(S, G_\mu, 0) = d_\mu(S, G_\lambda \times (-1, 1)^m, 0).$$

Hence

$$d_\mu(F_\mu, G_\mu, 0) \neq d_\mu(S, G_\mu, 0).$$

Define another mapping $S' : X\mu \to X_\mu$,

$$S'(x) = \phi_\lambda^*(Fx)) + x - P_\lambda(x).$$

We shall prove that $d_\mu(S, G_\mu, 0) = d_\mu(S', G_\mu, 0)$. Consider the homotopy

$$H(t, u) = tS(u) + (1-t)S'(u) = u - P_\lambda(u) + \phi_\lambda^*[tF(P_\lambda(u)) + (1-t)F(u)].$$

If $H(t, u) = 0$ for some $0 \leq t \leq 1$ and $u \in \bar{G}_\mu^{X_\mu}$, then $u = P_\lambda(u) \in X_\lambda$ implying $\phi_\lambda^*(F(u)) = F_\lambda(u) = 0$ and therefore $u \in G_\lambda \subset G_\mu$. By homoyopy invariance, we have

$$d_\mu(S, G_\mu, 0) = d_\mu(S', G_\mu, 0).$$

We can thus deduce that

$$d_\mu(F_\mu, G_\mu, 0) \neq d_\mu(S', G_\mu, 0).$$

Consider the homotopy $H : [0, 1] \times X_\mu \to X_\mu$,

$$H(t, u) = tS'(u) + (1-t)F_\mu(u).$$

By the homotopy invariance of the Brouwer degree, $H(t,u)=0$ for some $u\in\partial_{X_\mu}G_\mu\subset\partial_Y G$ and $t\in(0,1)$. Let $v\in X_\lambda$ be arbitrary, then

$$\begin{aligned}\langle H(t,u),v\rangle &= t\langle S'(u),v\rangle+(1-t)\langle F_\lambda(u),v\rangle\\ &= t\langle F(u),v\rangle+t\langle u-P_\lambda(u),v\rangle+(1-t)\langle F(u),v\rangle\\ &= t\langle F(u),v\rangle+(1-t)\langle F(u),v\rangle=0,\end{aligned}$$

implying $\langle F(u),v\rangle=0$. Moreover,

$$\begin{aligned}\langle H(t,u),u-P_\lambda(u)\rangle &= t\langle\phi_\lambda^*(F(u)),u-P_\lambda(u)\rangle+t\langle u-P_\lambda(u)),u-P_\lambda(u)\rangle\\ &+ (1-t)\langle F_\lambda(u),u-P_\lambda(u)\rangle\\ &= t||u-P_\lambda(u)||^2+(1-t)\langle F(u),u\rangle=0.\end{aligned}$$

Hence $\langle F(u),u\rangle\le 0$. □

Lemma 5.3.4 *Let $H:D_H\to Z$ be a strongly quasibounded (MOD) homotopy and $A\subset Y_0$ closed and bounded. If*

$$0\notin H([0,1]\times\bar{A}^{mod})\cap\overline{H([0,1]\times A)}^{mod},$$

then there exists $X_\lambda\in\Lambda$ such that

$$0\notin H_\mu([0,1]\times A_\mu)\ \textit{for all}\ X\mu\supset X_\lambda.$$

Proof. By contradiction, suppose that

$$\forall X_\lambda\in\Lambda,\exists X\mu\supset X_\lambda,\exists(t_\mu,a_\mu)\in[0,1]\times A_\mu;H_\mu(t_\mu,a_\mu)=0.$$

Define a set

$$V_\lambda=\{(t,a)\in[0,1]\times A\mid\langle H(t,a),a\rangle\le 0\text{ and }\langle H(t,a),v\rangle=0\text{ for all }v\in X_\lambda\},$$

which is non-empty for all λ. If $X_{\lambda_1},X_{\lambda_2},...,X_{\lambda_n}\in\Lambda$ and if $\cup_{i=1}^n X_{\lambda_i}\subset X_\lambda$, then

$$V_\lambda\subset\cap_{i=1}^n V_{\lambda_i}.$$

Therefore the family $\{V_\lambda\}$ has the finite intersection property. Denote τ the topology $||.||_{\mathbb{R}}\times\sigma(Y,Z_0)$. The set $\overline{V_\lambda}^\tau$ is bounded and τ-closed, which implies by Alaoglu's Theorem that it is τ-compact. Hence

$$\cap\overline{V_\lambda}^\tau\neq\emptyset.$$

Choose$(t_0,u_0)\in\cap\overline{V_\lambda}^\tau$. The space Y_0 is separable, we may denote $Y_0=\overline{\{y_1,y_2,...\}}$. Define $X_{\lambda_i}=sp\{y_1,y_2,...,y_i\}$, then we have $X_{\lambda_1}\subset X_{\lambda_2}\subset...\subset X_{\lambda_i}\subset....$ The space Z_0 is separable, then the

τ-topology in the set $\overline{[0,1]}^{\tau} \times A$ is metrizable ([86, p.782]. Denote d_τ this metric. For every $i = 1, 2, ...,$ we have $(t_0, u_0) \in \overline{V_{\lambda_i}}^{\tau}$, then we can choose a sequence $\{(t_n^{(i)}, u_n^{(i)})\}_n \subset V_{\lambda_i}$ such that

$$(\forall i \in \mathbb{N}), d_\tau\{(t_n^{(i)}, u_n^{(i)}), (t_0, u_0)\} < \frac{1}{n}.$$

Therefore $(t_n^{(n)}, u_n^{(n)}) \to (t_0, u_0)$ with respect to τ and

$$\langle H(t_n^{(n)}, u_n^{(n)}), u_n^{(n)} \rangle \leq 0, \tag{5.3.1}$$

$$\langle H(t_n^{(n)}, u_n^{(n)}), v \rangle = 0 \text{ for all } v \in X_{\lambda_n}. \tag{5.3.2}$$

Since H is strongly quasibounded, the sequence $\{H(t_n^{(n)}, u_n^{(n)})\}$ is bounded in Z by (5.3.1). Therefore we can choose a subsequence $\{H(t_{n'}^{(n')}, u_{n'}^{(n')})\}$ such that $H(t_{n'}^{(n')}, u_{n'}^{(n')}) \rightharpoonup z \in Z$. Moreover, by (5.3.2), we have

$$\langle \chi, v \rangle = 0 \text{ for all } v \in \cup_{n=1}^{\infty} X_{\lambda_n}.$$

Because $\cup_{n=1}^{\infty} X_{\lambda_n}$ is norm-dense in Y_0, we have

$$\langle \chi, v \rangle = 0 \text{ for all } v \in Y_0.$$

Therefore $\chi = 0$ as an element of the dual space Z. Moreover, Y_0 is $\sigma(Y, Z)$-dense in Y, which implies that

$$\langle \chi, v \rangle = 0 \text{ for all } v \in Y.$$

Consequently, we have

$$limsup \langle H(t_{n'}^{(n')}, u_{n'}^{(n')}), u_{n'}^{(n')} \rangle \leq 0 = \langle \chi, u_0 \rangle.$$

Since H is a (MOD) homotopy, we have $u_0 \in D_{H_t}$, $u_{n'}^{(n')} \to u_0(mod)$ in Y and
$H(t_{n'}^{(n')}, u_{n'}^{(n')}) \to H(t_0, u_0) = 0(mod)$ in Z for a subsequence. Therefor

$$0 \in H([0,1] \times \bar{A}^{mod}) \cap \overline{H([0,1] \times A)}^{mod},$$

which is a contradiction. □

The next lemma proves that the degree d_μ will stabilize when we go to the limit.

Lemma 5.3.5 *Let $F \in (MOD)$ be a strongly quasibounded and $G \subset Y_0$ open and bounded in Y_0. If*

$$0 \notin F(\overline{\partial_{Y_0} G}^{mod}) \cap \overline{F(\partial_{Y_0} G)}^{mod},$$

then there exists $X_\lambda \in \Lambda$ such that

$$0 \notin F_\mu(\partial_{X_\mu} G_\mu) \text{ and } d_\mu(F_\mu, G_\mu, 0) = constant \text{ for every } X_\mu \supset X_\lambda.$$

Proof. The first part follows immediately from Lemma 5.3.4.

For the second part, suppose, by contradiction, that

$$\forall X_\lambda \in \Lambda, \exists X_\mu \supset X_\lambda; d_\lambda(F_\lambda, G_\lambda, 0) \neq d_\mu(F_\mu, G_\mu, 0).$$

By Lemma 5.3.3, there exists $u_\lambda \in \partial_{Y_0} G$ such that

$$\langle F(u_\lambda), u_\lambda\rangle \leq 0 \text{ and } \langle F(u_\lambda), v\rangle = 0 \text{ for every } v \in X_\lambda.$$

Define a set

$$V_\lambda = u \in \partial_{Y_0} G \mid \langle F(u), u\rangle \leq 0 \text{ and } \langle F(u), v\rangle = 0 \text{ for all } v \in X_\lambda\},$$

which is non-empty. As in the proof of Lemma 5.3.4, we can deduce the existence of $u_0 \in \cap \overline{V_\lambda}^{\sigma(Y,Z_0)}$ and $\{u_n\} \subset \partial_{Y_0} G$ such that $u_n \to u_0 \in D_F(mod)$ in Y, $F(u_n) \to F(u_0)(mod)$ in Z for a subsequence and $F(u_0) = 0$.

Consequently, $0 \in F(\overline{\partial_{Y_0} G}^{mod}) \cap \overline{F(\partial_{Y_0} G)}^{mod}$, which is a contradiction. ☐

We can now define a degree function in the complementary system

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^{m,p(.)}(\Omega) & W^{-m,p'(.)}(\Omega) \\ H_0^{m,p(.)}(\Omega) & H^{-m,p'(.)}(\Omega) \end{pmatrix}$$

Definition 5.3.2 *Let $F \in \mathcal{F}$, $G \subset Y_0$ open and bounded in Y_0, $f \in Z_0$ and $f \notin F(\overline{\partial_{Y_0} G}^{mod}) \cap \overline{F(\partial_{Y_0} G)}^{mod}$. We then define*

$$d(F, G, f) = \lim_\lambda d_\lambda(F_\lambda - \phi_\lambda^*(f), G_\lambda, 0).$$

Theorem 5.3.1 *The mapping d in Definition 5.3.2 satisfies the conditions $(C_1) - (C_3)$. Any mapping $K \in \mathcal{F}^a$ satisfying*

$$\langle K(u), u\rangle > 0, \text{ when } u \neq 0, \text{ and } K(0) = 0$$

can be chosen as a normalising map.

Proof. It is enough to prove the conditions $(C_1) - (C_3)$ for $f = 0$, because $F - f \in \mathcal{F}$, $H - f \in \mathcal{H}$ and

$$d(F, G, f) = \lim_\lambda d_\lambda(F_\lambda - \phi_\lambda^*(f), G_\lambda, 0) = \lim_\lambda d_\lambda((F - f)_\lambda, G_\lambda, 0) = d(F - f, G, 0).$$

(C_1) If $d(F, G, 0) \neq 0$, then there exists $X_\lambda \in \Lambda$ such that $d_\mu(F_\mu, G_\mu, 0) \neq 0$ for all $X_\mu \supset X_\lambda$. Choose a sequence $\{v_{\mu_n}\}$ such that $v_{\mu_n} \in G_{\mu_n}, F_{\mu_n}(v_{\mu_n}) = 0$, $\dim X_{\mu_n} \to \infty$ and $\cup X_{\mu_n}$ is dense in Y_0. Chose a subsequence $\{v_{\mu_{n'}}\}$ such that $v_{\mu_{n'}} \rightharpoonup v \in Y$. Since $\langle F(v_{\mu_{n'}}), v_{\mu_{n'}}\rangle = 0$ and F is strongly quasibounded, we have $F(v_{\mu_{n'}}) \rightharpoonup \chi \in Z$ for a subsequence. We immediately see that $\langle \chi, w\rangle = 0$ for all $w \in X_{\mu_n}$ for every n. Therefore, by density, $\langle \chi, w\rangle = 0$ for all $w \in Y_0$, and by the $\sigma(Y,Z)$-density of Y_0 in the space Y, $\langle \chi, v\rangle = 0$. Hence $\chi = 0$ and

$$limsup\langle F(v_{\mu_{n'}}), v_{\mu_{n'}}\rangle = 0 = \langle \chi, v\rangle,$$

implying $v_{\mu_{n'}} \to v \in D_F(mod)$ in Y and $F(v_{\mu_{n'}}) \to F(v)(mod)$ in Z for a further subsequence. Therefore $0 \in F(\bar{G}^{mod}) \cap \overline{F(G)}^{mod}$.

(C_2) If $G_1, G_2 \subset G$ are open and bounded in Y_0, $G_1 \cap G_2 = \emptyset$ and

$$0 \notin F(\overline{\bar{G} \setminus (G_1 \cup G_2)}^{mod}) \cap \overline{F(\bar{G} \setminus (G_1 \cup G_2))}^{mod},$$

then, by Lemma 5.3.5, there exists $X_\lambda \subset \Lambda$ such that

$$0 \notin F_\mu(\bar{G}_\mu \setminus (G_{1,\mu} \cup G_{2,\mu})) \text{ for all } X_\mu \supset X_\lambda.$$

Hence

$$\begin{aligned} d(F,G,0) &= \lim_\lambda d_\lambda(F_\lambda, G_\lambda, 0) \\ &= \lim_\lambda [d_\lambda(F_\lambda, G_{1,\lambda}, 0) + d_\lambda(F_\lambda, G_{2,\lambda}, 0)] \\ &= d(F, G_1, 0) + d(F, G_2, 0). \end{aligned}$$

(C_3) Let $H \in \mathcal{H}$ and $G \subset Y_0$ be open and bounded in Y_0. Suppose that

$$0 \notin H([0,1] \times \overline{\partial_{Y_0} G}^{mod}) \cap \overline{H([0,1] \times \partial_{Y_0} G)}^{mod}.$$

By Lemma 5.3.4, there exists $X_\lambda \in \Lambda$ such that

$$0 \notin H_\mu([0,1] \times \partial G_\mu) \text{ for all } X_\mu \supset X_\lambda.$$

Consequently,

$$d_\mu(H_\mu(t,.), G_\mu, 0) = constant \text{ for all } t \in [0,1], \text{ when } X_\mu \supset X_\lambda.$$

Let $t_1, t_2 \in [0,1]$ be arbitrary. Then

$$d_\mu(H_\mu(t_1,.), G_\mu, 0) = d_\mu(H_\mu(t_2,.), G_\mu, 0),$$

and going to the limit we obtain

$$d(H(t_1,.), G, 0) = d(H(t_2,.), G, 0),$$

which means that $d(H(t,.), G, 0) = constant$ for all $t \in [0,1]$.

(C_4) Let $K \in \mathcal{F}^a$ a mapping satisfying

$$\langle K(u), u\rangle > 0, \text{ when } u \neq 0, \text{ and } K(0) = 0.$$

Suppose that $f \notin K(\overline{\partial_{Y_0} G}^{mod}) \cap \overline{K(\partial_{Y_0} G)}^{mod}$ and $f \in K(G)$. Let $u \in G$ be such that $K(u) = f$, and choose $X_\lambda \in \Lambda$ such that $u \in X_\lambda$. Then $\phi_\mu^*(f) \in K_\mu(G_\mu)$ for all $X_\mu \supset X_\lambda$. Moreover, $\langle K_\mu(v), v\rangle > 0$ for every $v \in X_\mu, v \neq 0$. By the basic properties of the Brouwer degree, we have $d_\mu(K_\mu, G_\mu, \phi_\mu^*(f)) = 1$ for all $X_\mu \supset X_\lambda$.
Hence $d(K, G, f) = 1$. □

5.3.2 Properties of the degree function

Using the conditions $(C_1) - (C_4)$ for the degree function, we can deduce some standard properties.

property 5.3.1 *Let $F, T \in \mathcal{F}^a, G \subset Y_0$ open and bounded in Y_0, $F/\partial_{Y_0} G = T/\partial_{Y_0} G$ and $f \in Z_0$. If $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and $f \notin F(\partial_Y G)$, then $d(F, G, f) = d(T, G, f)$.*

Proof. Define an affine homotopy $H : D_H \to Z$,

$$H(t, u) = tF(u) + (1-t)T(u),$$

which belongs ti the class $\mathcal{H}$ by Lemma 5.3.1. It is clear that

$$H([0,1] \times \partial_{Y_0} G) = F(\partial_{Y_0} G).$$

Since $f \notin F(\partial_Y G)$, we have $f \notin H([0,1] \times \partial_Y G)$. By homotopy invariance,

$$d(F, G, f) = d(T, G, f).$$

□

property 5.3.2 *Let $F \in \mathcal{F}$ and $G \subset Y_0$ is an open and bounded in Y_0. If $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, then $d(F, G, .)$ is constant on each open component in Z_0 of the open set $Z_0 \setminus F(\partial_Y G)$.*

Proof. Let $\Delta \subset Z_0 \setminus F(\partial_Y G)$ be an open component in Z_0 and $f_1, f_2 \in \Delta$ arbitrary. Then there exists a continuous curve $y : [0,1] \to Z_0$ such that $y(0) = f_1, y(1) = f_2$ and $y(t) \in \Delta$ for all $t \in [0,1]$. Therefore $y(t) \notin F(\partial_Y G)$. We see immediately that $F(u) - y(t) \in \mathcal{H}$ and $0 \notin F(\partial G) - y([0,1])$. By homotopy invariance,
$d(F, G, y(0)) = d(F, G, y(1))$ and we have the proof. □

property 5.3.3 *Let $F \in \mathcal{F}$, $G \subset Y_0$ open and bounded in Y_0 and $u_0 \in G$. Define a mapping $s : Y_0 \to Y_0, s(u) = u - u_0$. If $0 \notin F(\overline{\partial_{Y_0} G}^{mod}) \cap \overline{F(\partial_{Y_0} G)}^{mod}$, then*

$$d(F, G, 0) = d(Fos^{-1}, s(G), 0).$$

Proof. Choose $X_{\lambda_0} \in \Lambda$ such that $u_0 \in X_{\lambda_0}$. Now

$$d(F, G, 0) = \lim_{\lambda \geq \lambda_0} d_\lambda(F_\lambda, G_\lambda, 0).$$

By the properties of the Brouwer degree, we have

$$d_\lambda(F_\lambda, G_\lambda, 0) = d_\lambda(F_\lambda os^{-1}, s(G_\lambda), 0) = d_\lambda((Fos^{-1})_\lambda, (s(G))_\lambda), 0).$$

Moreover, it is easy to check that $Fos^{-1} \in \mathcal{F}$, $s(G) \subset Y_0$ is open and bounded in Y_0 and $0 \notin F(\overline{\partial_{Y_0} s(G)}^{mod}) \cap \overline{F(\partial_{Y_0} s(G))}^{mod}$. Therefore

$$d(F, G, 0) = \lim_{\lambda \geq \lambda_0} d_\lambda((Fos^{-1})_\lambda, (s(G))_\lambda), 0) = d(Fos^{-1}, s(G), 0).$$

□

Chapitre 6

Topological Degree methods for Partial Differential Operators in Generalized Sobolev Spaces

Abstract

The main aim of this chapter is to prove, by using the topological degree methods, the existence of solutions for nonlinear elliptic equation $Au = f$ where $Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u)$ is partial differential operators of general divergence form and $f \in W^{-m,p'(.)}(\Omega)$ with $p(x) \in (1, \infty)$.

6.1 Introduction

The degree theory construct by Berkovits and Mustonen [23, 25, 26] use the Leray-Schauder degree which can be applied to partial differential operators of general divergence form, i.e. to operators of the form

$$Au(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u).$$

If the functions $A_{\alpha}(x, \xi)$ satisfy the polynomial growth conditions with respect to $|\xi|$ and some analytical conditions, then the differential operator will generate a mapping defined in a Sobolev space,

$$A : W_0^{m,p}(\Omega) \to W^{-m,p'}(\Omega),$$

which belongs to the class (S_+). Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an open and bounded subset with a Lipschitz boundary $\partial\Omega$, A be a partial differential operator of general divergence form

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u))$$

defined on a subset of $W_0^{m,p(.)}(\Omega)$ and $f \in W^{-m,p'(.)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

The goal of the chapter is to prove the existence of solutions for nonlinear elliptic equations with boundary value problems of the form

$$\begin{cases} A(u) = f & \text{in} \quad \Omega \\ D^\alpha u(x) = 0 & \text{on} \quad \partial\Omega \text{ for } |\alpha| \leq m-1. \end{cases} \tag{6.1.1}$$

by applying the topological degree theory.

Our chapter is organized in the following way. The second section recalls some preliminary definitions and results about Generalized Lebesgue and Sobolev spaces, some classes of mappings of monotone type and defines a degree function in Sobolev spaces with variables exponents. The last section defines a new monotonicity class i.e. a class (MOD), presents some normalising maps and proves the existence of a solution for the problem (6.1.1) using the degree theory.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field [78].

6.2 Preliminary definitions and results

In the sequel, we consider a naturel number $N \geq 1$, an open and bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$ and we assume that $p(.)$ is a log-Hölder continuous exponent such that
$1 < p^- \leq p(x) \leq p^+ < \infty$. Under this assumption, we have :

1. Endowed with the Luxembourg norm, $L^{p(\cdot)}(\Omega)$ is a Banach space [63, Theorem 2.5], separable, reflexive [63, Corollary 2.7], uniformly convex and

$$[L^{p(\cdot)}(\Omega)]' = L^{p'(\cdot)}(\Omega).$$

2. The space $(W_0^{m,p(.)}(\Omega), \|\cdot\|_{m,p(.)})$ is a Banach space separable and reflexive and

$$[W_0^{m,p(.)}(\Omega)]' = W^{-m,p'(\cdot)}(\Omega).$$

3. If $q(.)$ is an exponent with $q^+ < \infty$, then $W_0^{m,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ (continuous embedding) if $q(.) \le p^*(.) = \frac{Np(.)}{N-p(.)}$. Moreover we have the compact embedding $W_0^{m,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$.

4. Norm convergence and modular convergence are equivalent.

We can prove, as in [84], the following lemmas

Lemma 6.2.1 *If* $\{u_n\} \subset L^{p(.)}(\Omega), \{v_n\} \subset L^{p'(.)}(\Omega), u_n \to u \in L^{p(.)}(\Omega)$ *in* $L^{p(.)}(\Omega)$ *and* $v_n \to v$ *a.e. and for the weak topology* $\sigma(L^{p'(.)}, L^{p(.)})$ *with* $v \in L^{p'(.)}(\Omega)$*, then* $u_n v_n \to uv$ *in* $L^1(\Omega)$.

Lemma 6.2.2 *If* $\{u_n\} \subset L^{p(.)}(\Omega), u_n \to u$ *a.e. with* $\in L^{p(.)}(\Omega)$ *and* $u_n \rightharpoonup u$ *in* $L^{p(.)}(\Omega)$ *and* $v \in L^{p'(.)}(\Omega)$*, then* $u_n v \to uv$ *in* $L^1(\Omega)$.

Lemma 6.2.3 *If* $\{u_n\} \subset L^{p(.)}(\Omega)$ *with* $u_n \to u$ *in* $L^{p(.)}(\Omega)$*, then*

$$\int_\Omega |u_n(x)|^{p(x)}\, dx \to \int_\Omega |u(x)|^{p(x)}\, dx.$$

Lemma 6.2.4 *(see also[56])*

(i) *If* $\{u_n\} \subset L^1(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^1(\Omega)$*,* $u_n, u \ge 0$ *a.e. and* $\int_\Omega u_n(x)\, dx \to \int_\Omega u(x)\, dx$ *, then* $u_n \to u$ *in* $L^1(\Omega)$.

(ii) *If* $\{u_n\} \subset L^1(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^1(\Omega)$*,* $\int_\Omega u_n(x)\, dx \to \int_\Omega u(x)\, dx$ *, and* $u_n(x) \ge -h(x)$ *a.e. for some* $h \in L^1(\Omega)$ *, then* $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 6.2.5 **(i)** *If* $\{u_n\} \subset L^{p(.)}(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^{p(.)}(\Omega)$*,* $u_n, u \ge 0$ *a.e. and* $|cu_n(x)|^{p(x)} \le h(x)$ *a.e. for some* $h \in L^1(\Omega)$ *and* $c > 0$ *then* $u_n \to u$ *in* $L^{p(.)}(\Omega)$.

(ii) *If* $\{u_n\} \subset L^{p(.)}(\Omega)$ *with* $u_n \to u \in L^{p(.)}(\Omega)$*, then there exists a subsequence* $\{u_{n'}\}, c > 0$ *and* $h \in L^1(\Omega)$ *such that* $u_{n'}(x) \to u(x)$ *a.e. and* $|cu_{n'}(x)|^{p(x)} \le h(x)$ *a.e.*

Lemma 6.2.6 **(i)** $(S_+) \cap (CONT) \subset (MOD)$.

(ii) $(MOD) \subset (PM)$.

Let $Y = W_0^{m,p(.)}(\Omega)$ and $Z = Y^* = W^{-m,p'(.)}(\Omega)$. We define the class $\mathcal{F}$ of *admissible mappings* and the class $\mathcal{H}$ of *admissible homotopies* as in paragraph 6.3.2 of Chapter 6

Theorem 6.2.1 *(see Remark 5.3.1) For* $F \in \mathcal{F}$ *,* $G \subset Y$ *open and bounded in* Y*,* $f \in Z$ *and* $f \notin F(\partial_Y G)$ *there exists an integer* $d(F, G, f)$ *(which is the degree function)satisfying the conditions :*

1. *(Existence) if* $d(F, G, f) \neq 0$ *, then* $f \in F(G)$*,*

2. *(Additivity) if $G_1, G_2 \subset G$ are open and bounded, $f \notin F(\bar{G} \setminus (G_1 \cup G_2))$, $G_1 \cap G_2 = \emptyset$, then*

$$d(F, G, f) = d(F, G_1, f) + d(F, G_2, f),$$

3. *(Homotopy invariance) if $H \in \mathcal{H}$,$f \in Z$ and $f \notin H([0,1] \times \partial_Y G)$,then*

$$d(H(t,.), G, f) = \textit{constant for all } t \in [0,1],$$

4. *(Normalization) There exists a normalising map $K \in \mathcal{F}^a$ such that if $f \in Z$,$f \notin K(\partial_Y G)$ and $f \in K(G)$, then*

$$d(K, G, f) = 1.$$

Any mapping $K \in \mathcal{F}^a$ satisfying

$$\langle K(u), u \rangle > 0, \textit{ when } u \neq 0, \textit{ and } K(0) = 0$$

can be chosen as a normalising map.

6.3 Differential Operators in Generalized Sobolev Spaces

6.3.1 Basic assumptions and technical lemmas

Let m be a positif integer. Denote $N_1 = \sum_{|\alpha| \leq m-1} 1$, $N_2 = \sum_{|\alpha|=m} 1$ and $N_0 = N_1 + N_2$.
Let $A_\alpha(x, \xi)$ be functions which satisfy the conditions :

(A_1) $A_\alpha : \Omega \times \mathbb{R}^{N_0} \to \mathbb{R}$ is a Caratheodory function for all $|\alpha| \leq m$.

(A_2) There exist an exponent $q(.)$, $(q(x) \in (1, \infty)$ with $q << p$ (i.e. $\inf_{x\in\Omega}(p(x) - q(x)) > 0$),
$a_\alpha \in L^{p'(.)}(\Omega)$ and constants $c_1, c_2 > 0$ such that :
$|A_\alpha(x, \xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2 \xi_\beta|^{p(x)-1} + c_1 \sum_{|\beta|<m} |c_2 \xi_\beta|^{\frac{p(x)}{q'(x)}}$ when $|\alpha| = m$,
$|A_\alpha(x, \xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2 \xi_\beta|^{\frac{q(x)}{p'(x)}} + c_1 \sum_{|\beta|<m} |c_2 \xi_\beta|^{p(x)-1}$ when $|\alpha| < m$,
for all $\xi \in \mathbb{R}^{N_0}$ and a.e. $x \in \Omega$.

(A_3) $\sum_{|\alpha|=m}(A_\alpha(x, \eta, \rho) - A_\alpha(x, \eta, \rho')).(\rho_\alpha - \rho'_\alpha) > 0$ a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}^{N_1}$ and $\rho, \rho' \in \mathbb{R}^{N_2}$, $\rho \neq \rho'$.

(A_4) There exist functions $b_\alpha \in L^{p'(.)}(\Omega)$ for $|\alpha| = m$, $b \in L^1(\Omega)$ constants $d_1, d_2 > 0$ and some fixed element $\phi \in W_0^{m,p(.)}(\Omega)$ such that :

$$\sum_{|\alpha|=m} A_\alpha(x, \xi)(\xi_\alpha - D^\alpha \phi(x)) \geq d_1 \sum_{|\alpha|=m} |d_2 \xi_\alpha|^{p(x)} - \sum_{|\alpha|=m} b_\alpha(x) \xi_\alpha - b(x)$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{N_0}$.

We denote in the following $\xi(u) = (D^\alpha u)_{|\alpha|\leq m}$ and $\eta(u) = (D^\alpha u)_{|\alpha|<m}$.
Define a mapping $A : W_0^{m,p(.)}(\Omega) \to W^{-m,p'(.)}(\Omega)$ by

$$\langle A(u), v\rangle = \sum_{|\alpha|\leq m} \int_\Omega A_\alpha(x, \xi(u)) D^\alpha v(x)\, dx \text{ for all } v \in W_0^{m,p(.)}(\Omega). \tag{6.3.1}$$

Assume that the conditions $(A_1) - (A_4)$ hold, then the operator A defined by (6.3.1) is continuous and of class (S_+) (the proof is the same as in [57, Proposition 27]). By Lemma 6.2.6, $T \in (MOD)$ and consequently A is pseudomonotone.

Lemma 6.3.1 *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$ is bounded, $\{t_n\} \subset [0, 1]$ and $\{\langle t_n A(u_n), u_n - \bar{u}\rangle\}$ is bounded for some $\bar{u} \in W_0^{m,p(.)}(\Omega)$, then the sequence $\{t_n A_\alpha(x, \xi(u_n))\}$ is bounded in $L^{p'(.)}(\Omega)$ for all $|\alpha| \leq m$.*

Proof. For $|\alpha| < m$ we use the fact that $q << p$, which implies that for every $\epsilon > 0$ there exists a constant $k(\epsilon)$ such that $|t|^{q(x)} \leq k(\epsilon)|\epsilon t|^{p(x)}$ for all $t > 0$. Therefore, by (A_2)
$t_n|A_\alpha(x, \xi(u_n))| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |k(\epsilon)|\epsilon c_2 D^\beta(u_n)|^{p(x)}|^{\frac{1}{p'(x)}} + c_1 \sum_{|\beta|<m} |c_2 D^\beta(u_n)|^{p(x)-1}$.
When ϵ is sufficiently small, $\|\epsilon c_2 D^\beta(u_n)\|_{p(.)} \leq 1$ uniformly for all $|\beta| \leq m$,

$$\||k(\epsilon)|\epsilon c_2 D^\beta(u_n)|^{p(x)}|^{\frac{1}{p'(x)}}\|_{p'(.)} \leq 1 + k(\epsilon) \int_\Omega |\epsilon c_2 D^\beta(u_n)|^{p(x)}\, dx$$

and

$$\||c_2 D^\beta(u_n)|^{p(x)-1}\|_{p'(.)} \leq 1 + \int_\Omega |c_2 D^\beta(u_n)|^{p(x)}\, dx$$

We conclude that

$$\begin{aligned}
\|t_n A_\alpha(x, \xi(u_n))\|_{p'(.)} &\leq \|a_\alpha(x)\|_{p'(.)} + c_1 \sum_{|\beta|=m} (1 + k(\epsilon))\|\epsilon c_2 D^\beta(u_n)\|_{p(.)} \\
&\quad + c_1 \sum_{|\beta|<m} (1 + \int_\Omega |c_2 D^\beta(u_n)|^{p(x)}\, dx) \\
&\leq cst.
\end{aligned}$$

To show the same property for $|\alpha| = m$, let $w = (w_\alpha) \in (L^{p(.)}(\Omega))^{N_2}$, by condition (A_3) we have

$$t_n \sum_{|\alpha|=m} (A_\alpha(x, \xi(u_n)) - A_\alpha(x, \eta(u_n), w))(D^\alpha(u_n) - w_\alpha) \geq 0.$$

for all $x \in \Omega$ and hence

$$\int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \xi(u_n))(w_\alpha - D^\alpha \bar{u})\, dx \leq$$

$$\langle t_n A(u_n), u_n - \bar{u}\rangle - \int_\Omega \sum_{|\alpha|<m} t_n A_\alpha(x, \xi(u_n))(D^\alpha u_n - D^\alpha \bar{u})\, dx$$

$$+ \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \eta(u_n), w)(w_\alpha - D^\alpha u_n)\, dx \tag{6.3.2}$$

The first term on the right remains bounded by the hypothesis of Lemma and the second one remains bounded by virtue of the previous argument. Moreover, by (A_2),

$$\|A_\alpha(x, \eta(u_n), w)\|_{p'(.)} \leq \|a_\alpha\|_{p'(.)} + c_1 \sum_{|\beta|=m} \||c_2 w_\beta|^{p(x)-1}\|_{p'(.)}$$
$$+ c_1 \sum_{\beta<m} \||k(\epsilon)|\epsilon c_2 D^\beta u_n|^{p(x)}|^{\frac{1}{p'(x)}}\|_{p'(.)}.$$

where

$$\||c_2 w_\beta|^{p(x)-1}\|_{p'(.)} \leq 1 + \int_\Omega |c_2 w_\beta|^{p(x)}\, dx \leq const$$

for all $|\beta| = 1$, since $w_\beta \in L^{p(.)}$. Moreover,

$$\||k(\epsilon)|\epsilon c_2 D^\beta u_n|^{p(x)}|^{\frac{1}{p'(x)}}\|_{p'(.)} \leq 1 + k(\epsilon) \int_\Omega |\epsilon c_2 D^\beta u_n|^{p(x)} \leq const,$$

when ϵ is made sufficiently small. Thus we have shown that $\{A_\alpha(x, \eta(u_n), w)\}$ is bounded in $L^{p'(.)}(\Omega)$, which implies that the third term on the right in (6.3.2) is also bounded. By the theorem of Banach-Steinhaus, the sequence $\{t_n A_\alpha(x, \xi(u_n))\}$ remains bounded in $L^{p'(.)}(\Omega)$ for every $|\alpha| = m$. □

Lemma 6.3.2 *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $u_n \to u \in W_0^{m,p(.)}$ in $W_0^{m,p(.)}$, $\{t_n\} \subset [0,1]$, $t_n \to t \in [0,1]$, $t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u))(\sigma(L^{p'(.)}, L^{p(.)})$ in $L^{p'(.)}(\Omega)$ and*

$$t_n \sum_{|\alpha|\leq m} A_\alpha(x, \xi(u_n)) D^\alpha u_n \to t \sum_{|\alpha|\leq m} A_\alpha(x, \xi(u)) D^\alpha u \text{ in } L^1(\Omega),$$

then $t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u))$ in $L^{p'(.)}(\Omega)$ for all $|\alpha| \leq m$.

Proof. When $|A_\alpha(x, \xi(u_n(x)))| \geq a_\alpha(x)$, we have from the condition (A_2) that

$$|\frac{t_n |A_\alpha(x, \xi(u_n(x)))|) - t_n a_\alpha(x)}{\lambda}|^{p'(x)}$$
$$\leq |\frac{t_n A_\alpha(x, \xi(u_n(x)))}{\lambda}| \times |\frac{t_n c_1}{\lambda} \sum_{|\beta|\leq m} |c_2 D^\beta u_n(x)|^{p(x)-1}|^{p'(x)-1}$$
$$\leq |\frac{t_n A_\alpha(x, \xi(u_n(x)))}{\lambda}| \times |\frac{t_n c_1}{\lambda} \sum_{|\beta|\leq m} |c_2 D^\beta u_n(x)|^{p(x)}|^{\frac{1}{p(x)}}$$
$$\leq \frac{c_2}{\lambda} \sum_{|\beta|\leq m} t_n |A_\alpha(x, \xi(u_n))) D^\beta u_n(x)|, \tag{6.3.3}$$

when $\lambda > 0$ is large enough. On the other hand , if $|A_\alpha(x, \xi(u_n(x)))| < a_\alpha(x)$, we have

$$|\frac{t_n|A_\alpha(x,\xi(u_n(x)))| - t_n a_\alpha(x)}{\lambda}|^{p'(x)} \leq |\frac{t_n a_\alpha(x)}{\lambda}|^{p'(x)}.$$

Therefore

$$|\frac{t_n|A_\alpha(x,\xi(u_n(x)))| - t_n a_\alpha(x)}{\lambda}|^{p'(x)} \leq |\frac{t_n a_\alpha(x)}{\lambda}|^{p'(x)} + \frac{c_2 t_n}{\lambda}\sum_{|\beta|\leq m}|A_\alpha(x,\xi(u_n(x)))D^\beta u_n(x)|$$

a.e. $x \in \Omega$, when $\lambda > 0$ is large enough. If $|\beta| \leq m-1$, then, by the compact imbedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, $D^\beta u_n \to D^\beta u$ in $L^{p(.)}$. By Lemma 6.2.1, $t_n A_\alpha(x,\xi(u_n))D^\beta u_n \to tA_\alpha(x,\xi(u))D^\beta u$ in $L^1(\Omega)$, when $|\beta| \leq m-1$.

Let $|\alpha|, |\beta| = m$ be arbitrary. Denote $\rho' = (0, 0, ..., 0, \frac{D^\beta u_n(x)}{\lambda}, 0, ..., 0)$, where $\frac{D^\beta u_n(x)}{\lambda}$ is the α^{the} coordinate of the vector ρ'. By condition (A_3),

$$t_n \sum_{|\gamma|=m}(A_\gamma(x,\eta(u_n),\rho(u_n)) - A_\gamma(x,\eta(u_n),\pm\rho'))(D^\gamma(u_n) \mp \rho'_\gamma) \geq 0,$$

and hence

$$t_n \sum_{|\gamma|=m}(A_\gamma(x,\xi(u_n))D^\gamma u_n) \pm \frac{t_n}{\lambda}A_\alpha(x,\eta(u_n),\pm\rho')D^\beta u_n \geq$$

$$t_n \sum_{|\gamma|=m}(A_\gamma(x,\eta(u_n),\pm\rho')D^\gamma(u_n) \mp \frac{t_n}{\lambda}A_\alpha(x,\xi(u_n))D^\beta u_n. \tag{6.3.4}$$

Let $|\gamma_1| = |\gamma_2| = m$ be arbitrary. Then

$$|A_{\gamma_1}(x,\eta(u_n),\pm\rho')D^{\gamma_2}u_n| \leq [a_{\gamma_1}(x) + c_1|\frac{c_2}{\lambda}D^\beta u_n|^{p(x)-1} + c_1\sum_{|\delta|\leq m-1}v_n^\delta(x)]|D^{\gamma_2}u_n|, \tag{6.3.5}$$

where $v_n^\delta(x) = |c_2 D^\delta u_n(x)|^{\frac{p(x)}{q'(x)}}$. Using Lemma 6.2.5, we obtain $D^\delta u_{n'} \to D^\delta u$ a.e. and $|c_2 D^\delta u_{n'}|^{p(x)} \leq h$ for all $|\delta| \leq m-1$ a.e. for some $h \in L^1(\Omega)$ and for some subsequence $\{u_{n'}\}$. We have v_n^δ and $v^\delta = |c_2 D^\delta u(x)|^{\frac{p(x)}{q'(x)}}$ belong to the space $L^{q'(.)}(\Omega)$, $v_{n'}^\delta \to v^\delta$ a.e. and $|v_{n'}^\delta|^{q'(x)} \leq |c_2 D^\delta u_{n'}|^{p(x)} \leq h$ a.e.

By using the Lemma 6.2.5 we can see that $v_{n'}^\delta \to v^\delta$ in $L^{q'(.)}$. Hence $v_{n'}^\delta \to v^\delta$ in $L^{p'(.)}$. By contradiction argument, $v_n^\delta \to v^\delta$ in $L^{p'(.)}$ and $D^{\gamma_2}u_n \rightharpoonup D^{\gamma_2}u$ in $L^{p(.)}$. Consequently, $a_{\gamma_1}D^{\gamma_2}u_n \to a_\alpha D^{\gamma_2}u$ in L^1 by Lemma 6.2.2 and $v_n^\delta D^{\gamma_2}u_n \to v^\delta D^{\gamma_2}u$ in L^1 by Lemma 6.2.1. Moreover,

$$\begin{aligned} |\frac{c_2}{\lambda}D^\beta u_n|^{p(x)-1}|D^{\gamma_2}u_n| &\leq |\frac{c_2}{\lambda}D^\beta u_n|^{p(x)-1}|D^\beta u_n| + |\frac{c_2}{\lambda}D^{\gamma_2}u_n|^{p(x)-1}|D^{\gamma_2}u_n| \\ &\leq \frac{2\lambda}{c_2}|\frac{c_2}{\lambda}D^\beta u_n|^{p(x)} + \frac{2\lambda}{c_2}|\frac{c_2}{\lambda}D^{\gamma_2}u_n|^{p(x)} \\ &\leq h_\beta + h_\gamma \in L^1(\Omega), \end{aligned}$$

when λ is large enough, because $D^\beta u_n \to D^\beta u$ and $D^{\gamma_2} u_n \to D^{\gamma_2} u$ in $L^{p(.)}$. Hence, by (6.3.5),

$$|A_{\gamma_1}(x, \eta(u_n), \pm\rho')D^{\gamma_2} u_n| \leq h \in L^1$$

and we obtain from (6.3.4)

$$t_n |A_\alpha(x, \xi(u_n))D^\beta u_n| \leq h_1 \in L^1.$$

Consequently, we can find that

$$t_n |A_\alpha(x, \xi(u_n))| - t_n a_\alpha(x) \to t|A_\alpha(x, \xi(u))| - t a_\alpha(x) \text{ in } L^{p'(.)}$$

by using lemma 6.2.5 and (6.3.3). Which implies that $t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u))$ in $L^{p'(.)}(\Omega)$ for every $|\alpha| \leq m$. $\square$

Lemma 6.3.3 *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $D^\alpha u_n \to D^\alpha u \in L^{p(.)}$ a.e. and $D^\alpha u_n \rightharpoonup D^\alpha u$ in $L^{p(.)}(\Omega)$ for all $|\alpha| \leq m$, $\{t_n\} \subset [0,1]$, $t_n \to t \in [0,1]$, $t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u))(\sigma(L^{p'(.)}, L^{p(.)})$ in $L^{p'(.)}(\Omega)$ for all $|\alpha| \leq m$ and $\langle t_n A(u_n), u_n\rangle \to \langle tA(u), u\rangle$, then*

$$t_n \sum_{|\alpha| \leq m} A_\alpha(x, \xi(u_n))D^\alpha u_n \to t \sum_{|\alpha| \leq m} A_\alpha(x, \xi(u))D^\alpha u$$

in $L^1(\Omega)$.

Proof. Denote $f_n(x) = t_n \sum_{|\alpha|=m} A_\alpha(x, \xi(u_n(x))D^\alpha u_n(x)$ and
$f(x) = t \sum_{|\alpha|=m} A_\alpha(x, \xi(u(x))D^\alpha u(x)$.
It is enough to prove that $f_n \to f$ in $L^1(\Omega)$ for a subsequence. By lemma 6.2.1, $a_\alpha D^\alpha u_n \to a_\alpha D^\alpha u$ in $L^1(\Omega)$, when $|\alpha| = m$. Consequently, for every $|\alpha| = m$, there exists $h_\alpha \in L^1(\Omega)$ such that $|a_\alpha(x) D^\alpha u_n(x)| \leq h_\alpha(x)$ a.e. for a subsequence. Denote $v_n^\beta(x) = |c_2 D^\beta u_n(x)|^{\frac{p(x)}{q'(x)}}$ and $v^\beta(x) = |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$, when $|\beta| < m$. On account of compactness of the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, we have $D^\beta u_n \to D^\beta u$ in $L^{p(.)}(\Omega)$, when $|\beta| \leq m-1$, which implies by lemma 6.2.3 and 6.2.5 that

$$\| |v_n^\beta|^{q'(x)} \|_1 \to \| |v^\beta|^{q'(x)} \|_1$$

and

$$|v_{n'}^\beta(x)|^{q'(x)} \to |v^\beta(x)|^{q'(x)} \text{ a.e.}$$

for some subsequence. By lemma 6.2.4, $|v_{n'}^\beta|^{q'(x)} \to |v^\beta|^{q'(x)}$ in $L^1(\Omega)$, and so for avery $|\beta| < m$ there must exist $h_\beta \in L^1$ such that

$$|v_n^\beta(x)|^{q'(x)} \leq h_\beta(x) \text{ a.e.}$$

Therefore $|v_n^\beta(x)| \leq |h_\beta(x)|^{\frac{1}{q'(x)}} \in L^{q'(\cdot)}(\Omega) \subset L^1(\Omega)$. Condition (A_3) implies that

$$t_n \sum_{|\alpha|=m} (A_\alpha(x, \eta(u_n), \rho(u_n)) - A_\alpha(x, \eta(u_n), \bar{0}))D^\alpha u_n \geq 0.$$

Consequently,

$$\begin{aligned} f_n(x) &\geq \sum_{|\alpha|=m} t_n A_\alpha(x, \eta(u_n), \bar{0}))D^\alpha u_n(x) \\ &\geq -t_n \sum_{|\alpha|=m} (|a_\alpha(x) D^\alpha u_n(x)| + c_1 \sum_{|\beta|<m} |c_2 D^\beta u_n(x)|^{\frac{p(x)}{q'(x)}}) \\ &\geq -\sum_{|\alpha|=m} (h_\alpha(x) + c_1 \sum_{|\beta|<m} |h_\beta(x)|^{\frac{1}{q'(x)}}) = -h(x) \in L^1(\Omega). \end{aligned}$$

Since $D^\alpha u_n \to D^\alpha u$ in $L^{p(\cdot)}$ for $|\alpha| < m$, we know from Lemma 6.2.1 that

$$t_n \sum_{|\alpha|<m} A_\alpha(x, \xi(u_n))D^\alpha u_n \to t \sum_{|\alpha|<m} A_\alpha(x, \xi(u))D^\alpha u$$

in $L^1(\Omega)$. Moreover, the assumption $\langle t_n A(u_n), u_n\rangle \to \langle tA(u), u\rangle$ implies that

$$\int_\Omega f_n(x)\, dx \to \int_\Omega f(x)\, dx.$$

By lemma 6.2.4, $f_n \to f$ in $L^1(\Omega)$, which completes the proof. □

Lemma 6.3.4 *Assume that the conditions* $(A_1) - (A_3)$ *hold. If the sequence* $\{u_n\} \subset W_0^{m,p(\cdot)}(\Omega)$, $u_n \rightharpoonup u$ *in* $W_0^{m,p(\cdot)}(\Omega)$, $\{t_n\} \subset [0,1]$, $t_n \to t$, $t_n A(u_n) \to z \in W^{-m,p'(\cdot)}(\Omega)(\sigma(W^{-m,p'(\cdot)}, W_0^{m,p(x)})$ *in* $W^{-m,p'(\cdot)}(\Omega)$ *and* $limsup \langle t_n A(u_n), u_n\rangle \leq \langle z, u\rangle$, *then*

$$\langle t_n A(u_n), u_n\rangle \to \langle z, u\rangle.$$

Proof. By lemma 6.3.1, the sequence $\{t_n A_\alpha(x, \xi(u_n)\}$ is bounded in $L^{p'(\cdot)}(\Omega)$ for all $|\alpha| \leq m$. We can thus assume that

$$t_n A_\alpha(x, \xi(u_n)) \to h_\alpha \in L^{p'(\cdot)}(\Omega)(\sigma(L^{p'(\cdot)}, L^{p(\cdot)})) \text{ in } L^{p'(\cdot)}(\Omega)$$

for $|\alpha| \leq m$. It is clear that

$$\langle z, w\rangle = \lim \langle t_n A(u_n), w\rangle = \lim \sum_{|\alpha|\leq m} \int_\Omega t_n A_\alpha(x, \xi(u_n))D^\alpha w(x)\, dx = \sum_{|\alpha|\leq m} \int_\Omega h_\alpha(x) D^\alpha w(x)\, dx$$

for all $w \in W_0^{m,p(\cdot)}(\Omega)$. By the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, we have

$$D^\alpha u_n \to D^\alpha u \text{ in } L^{p(\cdot)} \text{ for } |\alpha| < m.$$

Hence

$$\sum_{|\alpha|<m}\int_{\Omega} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n \to \sum_{|\alpha|<m}\int_{\Omega} h_\alpha(x)D^\alpha u.$$

Moreover, by the assumption,

$$\limsup \sum_{|\alpha|\le m}\int_{\Omega} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n \le \sum_{|\alpha|\le m}\int_{\Omega} h_\alpha D^\alpha u.$$

Therefore

$$\limsup \sum_{|\alpha|= m}\int_{\Omega} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n \le \sum_{|\alpha|= m}\int_{\Omega} h_\alpha D^\alpha u,$$

So that it is enough to prove that

$$\liminf \sum_{|\alpha|= m}\int_{\Omega} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n \ge \sum_{|\alpha|= m}\int_{\Omega} h_\alpha D^\alpha u.$$

Denote

$$\Omega_k = \{x\in\Omega / |D^\alpha u(x)| \le k \text{ for all } |\alpha| = m\}$$

and

$$E_k(x) = \begin{cases} 1 & \text{, when } x\in\Omega_k \\ 0 & \text{, otherwise.}\end{cases}$$

By condition (A_3) we have

$$\int_{\Omega}\sum_{|\alpha|=m} t_n[A_\alpha(x,\eta(u_n),E_k(x)\rho(u)) - A_\alpha(x,\xi(u_n))].[E_k(x)D^\alpha u - D^\alpha u_n] \ge 0.$$

Consequently,

$$\begin{aligned}\int_{\Omega}\sum_{|\alpha|=m} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n \;&\ge\; -\int_{\Omega}\sum_{|\alpha|=m} t_n A_\alpha(x,\eta(u_n),E_k(x)\rho(u))E_k(x)D^\alpha u \\ &+\; \int_{\Omega}\sum_{|\alpha|=m} t_n A_\alpha(x,\xi(u_n))E_k(x)D^\alpha u \\ &+\; \int_{\Omega}\sum_{|\alpha|=m} t_n A_\alpha(x,\eta(u_n),E_k(x)\rho(u))D^\alpha u_n. \qquad (6.3.6)\end{aligned}$$

By compact embedding, $D^\beta u_{n'} \to D^\beta u$ a.e. and $|c_2 D^\beta u_{n'}(x)|^{p(x)} \le h(x)$ a.e for some $h\in L^1(\Omega)$ and for $|\beta|\le m-1$ for some subsequence. Consequently,

$$(|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}})^{q'(x)} \le h(x)\text{a.e.}$$

By lemma 6.2.5,

$$|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}} \to |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$$

in $L^{q'(.)}(\Omega)$. Since $q << p$, we have

$$|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}} \to |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$$

in $L^{p'(.)}(\Omega)$. By (A_2),we obtain

$$|A_\alpha(x, \eta(u_n), E_k(x)\eta(u))| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2 k|^{p(x)-1} + c_1 \sum_{|\beta|\leq m-1} |c_2 D^\beta u_{n'}|^{\frac{p(x)}{q'(x)}}),$$

for $|\alpha| = m$. Now the right-hand side converges in $L^{p'(.)}(\Omega)$ and the left-hand side converges a.e.,so that it is easy to deduce that the left-hand side also converges in $L^{p'(.)}(\Omega)$. The first term on the right in (6.3.6) therefore tend towards

$$-\int_\Omega \sum_{|\alpha|=m} tA_\alpha(x, \eta(u), E_k(x)\eta(u)E_k(x)D^\alpha u,$$

and the third term on the right in (6.3.6) will tend towards

$$\int_\Omega \sum_{|\alpha|=m} tA_\alpha(x, \eta(u), E_k(x)\rho(u))D^\alpha u,$$

when n approaches infinity. Consequently,

$$\begin{aligned} \liminf \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \xi(u_n))D^\alpha u_n \quad &\geq \quad \liminf \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \xi(u_n))E_k(x)D^\alpha u \\ &+ \quad \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|=m} tA_\alpha(x, \eta(u), \bar{0})D^\alpha u \\ &= \quad \sum_{|\alpha|=m} \int_{\Omega_k} h_\alpha D^\alpha u + \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|=m} tA_\alpha(x, \eta(u), \bar{0})D^\alpha u, \end{aligned}$$

as $E_k(x)D^\alpha u \in L^{p(.)}(\Omega)$. Letting $k \to \infty$ we prove the lemma, since
$h_\alpha D^\alpha u \in L^1(\Omega)$ and $tA_\alpha(x, \eta(u), \bar{0})D^\alpha u \in L^1(\Omega)$ for $|\alpha| = m$. □

In the sequel we shall use the following well-known fact : if $u_n \rightharpoonup v$ in $L^{p(.)}(\Omega)$ and $u_n \to w$ a.e. in Ω, then $v = w$ a.e. in Ω.

Lemma 6.3.5 *Assume that the conditions* $(A_1) - (A_4)$ *hold. If the sequence* $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $u_n \rightharpoonup u \in W_0^{m,p(.)}(\Omega)$ *in* $W_0^{m,p(.)}(\Omega)$, $A(u_n) \to z \in W^{-m,p'(.)}(\Omega)(\sigma(W^{-m,p'(.)}, W_0^{m,p(.)})$ *in* $W^{-m,p'(.)}(\Omega)$ *and* $limsup\langle A(u_n), u_n\rangle \leq \langle z, u\rangle$, *then* $u_{n'} \to u$ *in* $W_0^{m,p(.)}(\Omega)$ *for some subsequence.*

Proof. We deduce as in [66] that $D^\alpha u_n(x) \to D^\alpha u(x)$ a.e. for $|\alpha| \leq m$ and for some subsequence. According to lemma 6.3.1 we may assume that

$$A_\alpha(x, \xi(u_n)) \to w_\alpha(x)(\sigma(L^{p'(.)}, L^{p(.)})) \text{ in } L^{p'(.)}$$

for every $|\alpha| \leq m$. Since

$$A_\alpha(x, \xi(u_n)) \to A_\alpha(x, \xi(u)) \text{ a.e.},$$

we know that $w_\alpha(x) = A_\alpha(x, \xi(u))$ a.e. A is pseudomonotone.
Hence

$$z = A(u) \text{ and } \langle A(u_n), u_n \rangle \to \langle A(u), u \rangle.$$

By lemma 6.3.3,

$$\sum_{|\alpha| \leq m} A_\alpha(x, \xi(u_n)) D^\alpha u_n \to \sum_{|\alpha| \leq m} A_\alpha(x, \xi(u)) D^\alpha u$$

in $L^1(\Omega)$, and by condition (A_4),

$$\begin{aligned} d_1 \sum_{|\alpha|=m} |d_2 D^\alpha u_n(x)|^{p(x)} &\leq \sum_{|\alpha|=m} A_\alpha(x, \xi(u_n(x)) D^\alpha u_n(x) - \sum_{|\alpha|=m} A_\alpha(x, \xi(u_n(x)) D^\alpha \varphi(x) \\ &+ \sum_{|\alpha|=m} b_\alpha(x) D^\alpha u_n(x) + b(x). \end{aligned}$$

The right hand side converges in $L^1(\Omega)$ in accordance with Lemma 6.2.1. Lemma 6.2.5 implies $D^\alpha u_n \to D^\alpha u$ in $L^{p(.)}(\Omega)$ for all $|\alpha| = m$. By compact embedding, $D^\alpha u_n \to D^\alpha u$ in $L^{p(.)}$, when $|\alpha| < m$, which completes the proof. □

Theorem 6.3.1 *If the conditions $(A_1) - (A_4)$ hold, then the mapping A defined by* (6.3.1) *belong to the class $\mathcal{F}^a$.*

Proof. Strong quasiboundedness and condition b) of class $\mathcal{F}^a$ follow immediately from Lemma 6.3.1. Lemma 6.3.4 implies condition c), and condition d) follows from lemmas 6.3.4, 6.3.3 and 6.3.2. Suppose that $\{u_n\} \subset W_0^{m,p(.)}(\Omega), u_n \rightharpoonup u$ in $W_0^{m,p(.)}(\Omega), A(u_n) \to z \in Z(\sigma(W^{-m,p'(.)}, W_0^{m,p(.)}))$ in $W^{-m,p'(.)}(\Omega)$ and $\limsup_{n\to\infty} \langle A(u_n), u_n \rangle \leq \langle z, u \rangle$.
By pseudomonotonicity, $z = A(u)$ and $\langle A(u_n), u_n \rangle \to \langle A(u), u \rangle$. If (A_4) holds, then, by Lemma 6.3.5, $u_n \to u$ in $W_0^{m,p(.)}(\Omega)$ for some subsequence. Choosing $t_n = 1$ in Lemma 6.3.3 and Lemma 6.3.2 we may deduce that $A(u_n) \to A(u)$ (mod) in $W^{-m,p'(.)}(\Omega)$. Hence $A \in (MOD)$. □

6.3.2 Normalising maps

Let $Y = W_0^{m,p(.)}(\Omega)$, $Z = W^{-m,p'(.)}(\Omega)$. We start with an abstract existence theorem.

Theorem 6.3.2 *Let $G \subset Y$ be open and bounded in Y, $0 \in G$, $f \in Z$ and $F \in \mathcal{F}^a$. Suppose that there exists a normalising map $K \in \mathcal{F}^a$ such that $K(0) = 0$ and $\langle K(u), u \rangle \geq b > 0$ for all $u \in \partial_Y G$. Choose a* *constant* $a \geq 0$ *such that*

$$a \leq \inf_{u \in \partial_Y G} \frac{\langle K(u), u \rangle}{\|K(u)\|}.$$

If $\langle F(u) - f, u \rangle + \|F(u) - f\|a > 0$ *for all* $u \in \partial_Y G$*, then* $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. Since $\langle K(u), u \rangle \geq b > 0$ for all $u \in \partial_Y G$, it is clear that $0 \notin \overline{K(\partial_Y G)}$. Define by

$$H(t, u) = tF(u) + (1 - t)K(u).$$

a homotopy which belong to the class $\mathcal{H}$ by Lemma 5.3.1. We show that $H(t, u) \neq 0$ for all $t \in [0, 1]$, $u \in \partial_Y G$. If $0 \in H([0, 1] \times \partial_Y G)$, then

$$tF(u) + (1 - t)K(u) = 0$$

for some $u \in \partial_Y G$ and $t \in [0, 1]$. It is clear that $t \neq 0$. Thus

$$t||F(u)|| = (1 - t)||K(u)||,$$

implying

$$1 - \frac{1}{t} = -\frac{||F(u)||}{||K(u)||}.$$

On the other hand,

$$t\langle F(u), u \rangle + (1 - t)\langle K(u), u \rangle = 0,$$

and therefore

$$\langle F(u), u \rangle = (1 - \frac{1}{t})\langle K(u), u \rangle = -||F(u)||\frac{\langle K(u), u \rangle}{||K(u)||} \leq -||F(u)||a,$$

which is a contradiction. Hence $H(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_Y G$. By homotopy invariance,

$$d(F, G, 0) = f(K, G, 0).$$

Since $0 \in K(G)$, we have $d(K, G, 0) = 1$. □

Corollary 6.3.1 *Let* $G \subset Y$ *be open and bounded in* Y*,* $\bar{u} \in G$*,* $f \in Z$ *and* $F \in \mathcal{F}^a$*. Suppose that there exists a normalising map* $K \in \mathcal{F}^a$ *such that* $\tilde{K}(u) = K(u + \bar{u}) - K(\bar{u})$ *is also a normalising map in the class* $\mathcal{F}^a$ *and*

$$\langle K(u) - K(\bar{u}), u - \bar{u} \rangle \geq b > 0 \text{ for all } u \in \partial_Y G.$$

Choose a constant $a \geq 0$ *such that*

$$a \leq \inf_{u \in \partial_Y G} \frac{\langle K(u) - K(\bar{u}), u - \bar{u} \rangle}{\|K(u) - K(\bar{u})\|}.$$

If $\langle F(u) - f, u - \bar{u} \rangle + \|F(u) - f\|a > 0$ *for all* $u \in \partial_Y G$*, then* $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. It follows from the above assumptions that the degree $d(F, G, 0)$ is defined. Define $s(u) = u - \bar{u}$. Then, by property 5.3.3,

$$d(F, G, 0) = d(Fos^{-1}, s(G), 0).$$

Let $u \in \partial_Y s(G)$ be arbitrary. Denote $u = u' - \bar{u}$, where $u' \in \partial_Y G$.
Now $\langle \tilde{K}(u), u\rangle = \langle K(u') - K(\bar{u}), u' - \bar{u}\rangle \geq b > 0$ and

$$\frac{\langle \tilde{K}(u), u\rangle}{||\tilde{K}(u)||} = \frac{\langle K(u') - K(\bar{u}), u - \bar{u}\rangle}{\|K(u') - K(\bar{u})\|} \geq a \geq 0.$$

Moreover,

$$\langle Fos^{-1}(u), u\rangle + ||Fos^{-1}(u)||a = \langle F(u'), u' - \bar{u}\rangle + ||F(u')||a.$$

Therefore the assumptions of Theorem 6.3.2 hold. Hence

$$d(Fos^{-1}(u), s(G), 0) = 1.$$

□

Next we shall present two mappings of class $\mathcal{F}^a$, which can be used as normalising maps.

Theorem 6.3.3 *The mapping* $K : W_0^{m,p(.)}(\Omega) \to W^{-m,p'(.)}(\Omega)$ *defined by*

$$\langle Ku, v\rangle = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)-1} sgn D^\alpha u(x) D^\alpha v(x)\, dx \text{ for all } v \in W_0^{m,p(.)}(\Omega),$$

belongs to the class $\mathcal{F}^a$, $\langle K(u), u\rangle > 0$ *for* $u \in W_0^{m,p(.)}(\Omega)$, $u \neq 0$ *and* $K(0) = 0$.

Proof. Denote $A_\alpha(x, \eta, \rho) = |\rho_\alpha|^{p(x)-1} sgn \rho_\alpha$ when $|\alpha| = m$ and $A_\alpha(x, \eta, \rho) = 0$ when $|\alpha| \leq m - 1$. The mappings A_α are clearly continuous with respect to η and ρ. Hence condition (A_1) is satisfied. It is obvious that condition (A_2) holds. Since the function $|\rho_\alpha|^{p(x)-1} sgn \rho_\alpha$ is strictly increasing, we obtain condition (A_3). Condition (A_4) is reduced to

$$\sum_{|\alpha|=m} A_\alpha(x, \xi).\xi_\alpha \geq \sum_{|\alpha|=m} |\xi_\alpha|^{p(x)},$$

when we choose $\varphi \equiv 0$,$b_\alpha = b = 0$ and $d_1 = d_2 = 1$. Moreover,

$$\langle K(u), u\rangle = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)-1} sgn D^\alpha u(x) D^\alpha u(x)\, dx = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)}\, dx \geq 0,$$

and the equality holds if and only if $D^\alpha u(x) = 0$ a.e. for every $|\alpha| = m$, which implies that $u \equiv 0$. □

Theorem 6.3.4 *The mapping, so-called $p(.)$-Laplacian, $K : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega)$ defined by*

$$K(u) = -\Delta_{p(.)}(u) = -div(|\nabla u(x)|^{p(x)-2}\nabla u(x)).$$

i.e.

$$\langle Ku, v\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x).\nabla v(x)\,dx \text{ for all } v \in W_0^{1,p(.)}(\Omega),$$

belongs to the class $\mathcal{F}^a$, $\langle K(u), u\rangle > 0$ for $u \in W_0^{1,p(.)}(\Omega)$, $u \neq 0$ and $K(0) = 0$.

Proof. Denote $A_i(x, \eta, \rho) = |\rho|^{p(x)-2}\rho_i, i = 1, 2, ..., N$, when $\rho \neq \bar{0}$, $A_i(x, \eta, \bar{0}) = 0$. Since $|A_i(x, \eta, \rho)| \leq |\rho|^{p(x)-1}$, the function A_i is continuous with respect to η and ρ. Hence condition (A_1) is satisfied. Condition (A_2) is easily seen to hold. Let $f(t) = |t|^{p(x)-1}/t$, when $t \neq 0$, $f(0) = 0$. The function $f(t)t$ is strictly increasing in $[0, \infty)$, which gives

$$(f(|\rho|)|\rho| - f(|\rho'|)|\rho'|)(|\rho| - |\rho'|) > 0,$$

when $\rho, \rho' \in \mathbb{R}^{N_2}, |\rho| \neq |\rho'|$. Hence

$$\sum_{i=1}^{N}[f(|\rho|)\rho_i{}^2 + f(|\rho'|)\rho_i'^2] - [f(|\rho|) + f(|\rho'|)]|\rho||\rho'| > 0.$$

By the Cauchy-Schwarz inequality,$|\rho||\rho'| \geq \sum_{i=1}^{N}\rho_i\rho_i'$, implying

$$\sum_{i=1}^{N}[f(|\rho|)\rho_i{}^2 + f(|\rho'|)\rho_i'^2] - \sum_{i=1}^{N}[f(|\rho|) + f(|\rho'|)]\rho_i\rho_i' > 0.$$

Consequently,

$$\sum_{i=1}^{N}[f(|\rho|)\rho_i - f(|\rho'|)\rho_i'][\rho_i - \rho_i'] > 0,$$

when $|\rho| \neq |\rho'|$. If $|\rho| = |\rho'|$, we have equality in the Cauchy-Schwarz inequality only if $\rho = \rho'$, and hence strict inequality follows. Therefore condition (A_3) holds. Condition (A_4) follows as in Theorem 6.3.3. Moreover,

$$\langle K(u), u\rangle = \int_\Omega |\nabla u(x)|^{p(x)}\,dx \geq 0,$$

and equality holds only if $\nabla u(x) = 0$ a.e., which implies that $u = 0$ a.e. □

The previous theorems imply the existence of a normalising map in Sobolev space with variables exponents which satisfy the conditions of the previous section. We shall equip the space $Y = W_0^{m,p(.)}(\Omega)$ with the norm

$$||u||_Y = \sqrt{\sum_{|\alpha|=m} ||D^\alpha u||_{p(.)}^2}.$$

Let $G \subset W_0^{m,p(.)}(\Omega)$ be open and bounded in $W_0^{m,p(.)}(\Omega)$ and $0 \in G$. Since the set $\partial_Y G$ is closed, we have

$$\inf_{u \in \partial_Y G} ||u||_Y > 0.$$

We may therefore choose a constant $c \in \mathbb{R}$ such that

$$0 < c < \frac{1}{\sqrt{N_2}} \inf_{u \in \partial_Y G} ||u||_Y.$$

Moreover, we have

$$c < \frac{1}{\sqrt{N_2}} \sqrt{\sum_{|\alpha|=m} ||D^\alpha u||_{p(.)}^2} - \epsilon \leq \frac{1}{\sqrt{N_2}} \sqrt{N_2 \max_{|\alpha|=m} \{||D^\alpha u||_{p(.)}^2\}} - \epsilon = \max_{|\alpha|=m} \{||D^\alpha u||_{p(.)}\} - \epsilon$$

for some $\epsilon > 0$ and for all $u \in \partial_Y G$. Define a mapping $K \in \mathcal{F}^a$,

$$Ku(x) = \sum_{|\alpha=m} (-1)^{|\alpha|} D^\alpha (|\frac{D^\alpha u(x)}{c}|^{p(x)-1} sgn D^\alpha u(x)).$$

Let $u \in \partial_Y G$ and $||D^{\alpha_0} u||_{p(.)} = \max_{|\alpha|=m} \|D^\alpha u\|_{p(.)}$. Now

$$\begin{aligned}
\langle K(u), u \rangle &= c \sum_{|\alpha|=m} \int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)-1} |\frac{D^\alpha u(x)}{c}| \, dx \\
&= c \sum_{|\alpha|=m} \int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)} \, dx \\
&\geq c \int_\Omega |\frac{D^{\alpha_0} u(x)}{||D^{\alpha_0} u||_{p(.)} - \epsilon}|^{p(x)} \, dx \\
&\geq c.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|K(u)\|_Z = \sup_{\|v\|_Y \leq 1} \langle K(u), v \rangle &= \sup_{\|v\|_Y \leq 1} \sum_{|\alpha|=m} \int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)-1} sgn D^\alpha u(x) D^\alpha v(x) \, dx \\
&\leq \sup_{\|v\|_Y \leq 1} 2 \sum_{|\alpha|=m} \| |\frac{D^\alpha u(x)}{c}|^{p(x)-1} \|_{p'(.)} \|D^\alpha v\|_{p(.)} \\
&\leq 2 \sum_{|\alpha|=m} \| |\frac{D^\alpha u(x)}{c}|^{p(x)-1} \|_{p'(.)}
\end{aligned}$$

Consequently,

$$\frac{\langle K(u), u \rangle}{\|K(u)\|_Z} \geq \frac{c \sum_{|\alpha|=m} \int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)} \, dx}{2 \sum_{|\alpha|=m} \| |\frac{D^\alpha u(x)}{c}|^{p(x)-1} \|_{p'(.)}}.$$

If $\| |\frac{D^\alpha u(x)}{c}|^{p(x)-1} \|_{p'(.)} \leq 1$ for all $|\alpha| = m$, then

$$\frac{\langle K(u), u \rangle}{\|K(u)\|_Z} \geq \frac{c}{2N_2} \int_\Omega |\frac{D^{\alpha_0} u(x)}{c}|^{p(x)} \geq \frac{c}{2N_2}.$$

If

$$\max_{|\alpha|=m} \||\frac{D^{\alpha}u(x)}{c}|^{p(x)-1}\|_{p'(.)} = \||\frac{D^{\alpha_1}u(x)}{c}|^{p(x)-1}\|_{p'(.)} > 1$$

for some $|\alpha_1| = m$, then we obtain

$$\int_{\Omega} |\frac{D^{\alpha_1}u(x)}{c}|^{p(x)} \, dx = \int_{\Omega} ||\frac{D^{\alpha_1}u(x)}{c}|^{p(x)-1}|^{p'(x)} \, dx \geq \||\frac{D^{\alpha_1}u(x)}{c}|^{p(x)-1}\|_{p'(.)}.$$

Hence

$$\frac{\langle K(u), u\rangle}{\|K(u)\|_Z} \geq \frac{c\sum_{|\alpha|=m}\int_{\Omega}|\frac{D^{\alpha}u(x)}{c}|^{p(x)}\, dx}{2\sum_{|\alpha|=m}\||\frac{D^{\alpha}u(x)}{c}|^{p(x)-1}\|_{p'(.)}} \geq \frac{c\int_{\Omega}|\frac{D^{\alpha_1}u(x)}{c}|^{p(x)}\, dx}{2N_2\||\frac{D^{\alpha_1}u(x)}{c}|^{p(x)-1}\|_{p'(.)}} \geq \frac{c}{2N_2}.$$

In other words, we have constructed a normalising map K for which the conditions of this section hold. For example, if $G = B_R(0)$, we may choose $c = \frac{R}{2\sqrt{N_2}}$, $b = \frac{R}{2\sqrt{N_2}}$,$a = \frac{R}{4N_2\sqrt{N_2}}$ and

$$Ku(x) = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(|\frac{D^{\alpha}u(x)}{c}|^{p(x)-1}sgnD^{\alpha}u(x))$$

in Theorem 6.3.2. While, if $G = B_R(\bar{u})$ for some $\bar{u} \in W_0^{m,p(.)}$, we may choose a, b and c as above and

$$Ku(x) = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(|\frac{D^{\alpha}u(x) - D^{\alpha}\bar{u}}{c}|^{p(x)-1}sgnD^{\alpha}u(x))) + \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(|\frac{D^{\alpha}\bar{u}(x)}{c}|^{p(x)-1}sgnD^{\alpha}\bar{u}(x))$$

in Corollary 6.3.1.

6.3.3 Existence results

Theorem 6.3.5 *Assume that the conditions $(A_1) - (A_4)$ hold. Define the mapping A as in* (6.3.1). *Let $f \in W^{-m,p'(.)}(\Omega)$. If*

$$\liminf_{\|u\|_{m,p(.)}\to\infty} \langle A(u) - f, u - \bar{u}\rangle \geq 0 \textit{ for some } \bar{u} \in W_0^{m,p(.)}(\Omega), \tag{6.3.7}$$

then the problem (6.1.1) *is almost solvable, i.e., $f \in \overline{A(W_0^{m,p(.)}(\Omega))}$. If*

$$\liminf_{\|u\|_{m,p(.)}\to\infty} \langle A(u) - f, u - \bar{u}\rangle > 0 \textit{ for some } \bar{u} \in W_0^{m,p(.)}(\Omega), \tag{6.3.8}$$

then the problem (6.1.1) *is solvable, i.e., $f \in A(W_0^{m,p(.)}(\Omega))$.*

Proof. Suppose first that (6.3.7) holds. By theorem 6.3.1, $A \in \mathcal{F}^a$. If

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R}((\langle A(u) - f, u - \bar{u}\rangle + \|A(u) - f\|\frac{R}{4N_2\sqrt{N_2}}) \leq 0 \text{ for all } R > 0,$$

then, by (6.3.7), there exists a sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$ such that $\|u_n\|_{m,p(.)} \to \infty$ and $\|A(u_n) - f\| \to 0$. Hence $f \in \overline{A(W_0^{m,p(.)}(\Omega))}$.Suppose that

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R} (\langle A(u) - f, u - \bar{u}\rangle + \|A(u) - f\| \frac{R}{4N_2\sqrt{N_2}}) > 0 \text{ for some } R > 0.$$

As indicated in the previous section, there exists a normalising map K satisfying the assumptions of Corollary 6.3.1 with $b = \frac{R}{2\sqrt{N_2}}$ and $a = \frac{R}{4N_2\sqrt{N_2}}$. Denote

$$B_R(\bar{u}) = \{u \in W_0^{m,p(.)}(\Omega)/\|u - \bar{u}\|_Y \leq R\}.$$

By Corollary 6.3.1, $d(A, B_R(\bar{u}), f) = 1$. By property (1) of the degree function, $f \in A(B_R(\bar{u})) \subset \overline{A(W_0^{m,p(.)}(\Omega))}$. If (6.3.8) holds, then we clearly have

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R} (\langle A(u) - f, u - \bar{u}\rangle + \|A(u) - f\| \frac{R}{4N_2\sqrt{N_2}}) > 0 \text{ for some } R > 0,$$

and proceeding as above we obtain $f \in A(W_0^{m,p(.)}(\Omega))$. □

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